ON A SUM OF MELHAM AND ITS VARIANTS

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Abstract. We prove a general expansion formula for a sum of powers of Fibonacci numbers, as considered by Melham, as well as some extensions

1. Introduction

Wiemann and Cooper [2] report about some conjectures of Melham [1] related to the sum

$$\sum_{k=1}^{n} F_{2k}^{2m+1}.$$

We don't know what these conjectures are, but an expansion for $m = 2$ is cited:

$$\sum_{k=1}^{n} F_{2k}^{5} = \frac{1}{L_1 L_3 L_5}\Big[4F_{2n+1}^{5} - 15F_{2n+1}^{3} + 25F_{2n+1} - 14\Big].$$

In this paper, we will prove the general formula

$$\sum_{k=1}^{n} F_{2k}^{2m+1} = \sum_{l=0}^{m} \lambda_{m,l} F_{2n+1}^{2l+1} + C_m,$$

with

$$\lambda_{m,l} = 5^{l-m}(-1)^{m-l}\frac{1}{2l+1}\sum_{j=0}^{m-l}\binom{2m+1}{j}\binom{m-j+l}{m-j-l}\frac{2m-2j+1}{L_{2m-2j+1}}$$

and

$$C_m = \frac{1}{5^m}\sum_{j=0}^{m}(-1)^{j-1}\binom{2m+1}{j}\frac{F_{2m-2j+1}}{L_{2m-2j+1}}.$$

(We use Fibonacci and Lucas numbers, as usual.)

In September 2008, after I obtained these results, I learnt from C. Cooper that somebody anticipated them, by 2 months! So I decided to go ahead and find more results. These are the evaluations of

$$\sum_{k=0}^{n} F_{2k+\delta}^{2m+\varepsilon},$$

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for $\delta, \varepsilon \in \{0,1\}$, as well as the evaluations of the corresponding sums for Lucas numbers. All in all these are 8 formulæ of the Melham type. At the end, we will apply them to compute *iterates* of the original Melham sum.

We decided to present the original proof of the Melham sum, since the remaining instances are done in a similar style, whence we present then only some key steps.

2. Proof

We will make extensive use of the Binet formulæ

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2};$$

note that $\beta = -1/\alpha$.

We need a Lemma.

Lemma 1.

$$x^{2k+1} + x^{-2k-1} = \sum_{l=0}^{k} (-1)^{k-l} \frac{2k+1}{2l+1} \binom{k+l}{k-l} \big(x + \tfrac{1}{x}\big)^{2l+1}. \tag{1}$$

The proof will be by induction, the instance $k = 0$ being clear. We compute

$$\begin{aligned}
x^{2k+3} + x^{-2k-3} &= (x^{2k+1} + x^{-2k-1})(x^2 + x^{-2}) - (x^{2k-1} + x^{-2k+1}) \\
&= (x^2 + x^{-2}) \sum_{l=0}^{k} (-1)^{k-l} \frac{2k+1}{2l+1} \binom{k+l}{k-l} (x + \tfrac{1}{x})^{2l+1} \\
&\qquad - \sum_{l=0}^{k-1} (-1)^{k-1-l} \frac{2k-1}{2l+1} \binom{k-1+l}{k-1-l} (x + \tfrac{1}{x})^{2l+1} \\
&= \sum_{l=0}^{k} (-1)^{k-l} \frac{2k+1}{2l+1} \binom{k+l}{k-l} (x + \tfrac{1}{x})^{2l+3} \\
&\qquad - 2 \sum_{l=0}^{k} (-1)^{k-l} \frac{2k+1}{2l+1} \binom{k+l}{k-l} (x + \tfrac{1}{x})^{2l+1} \\
&\qquad - \sum_{l=0}^{k-1} (-1)^{k-1-l} \frac{2k-1}{2l+1} \binom{k-1+l}{k-1-l} (x + \tfrac{1}{x})^{2l+1}.
\end{aligned}$$

We compute the coefficient of $(x + \frac{1}{x})^{2l+1}$:

$$(-1)^{k-l-1} \frac{2k+1}{2l-1} \binom{k+l-1}{k-l+1} - 2(-1)^{k-l} \frac{2k+1}{2l+1} \binom{k+l}{k-l} - (-1)^{k-1-l} \frac{2k-1}{2l+1} \binom{k-1+l}{k-1-l}$$

$$= (-1)^{k+1-l}\frac{2k+3}{2l+1}\binom{k+1+l}{k+1-l},$$

as desired. □

And now comes the big computation:

$$\begin{aligned}
\sum_{k=1}^{n} F_{2k}^{2m+1} &= \frac{1}{5^m\sqrt{5}}\sum_{k=1}^{n}(\alpha^{2k}-\beta^{2k})^{2m+1}\\
&= \frac{1}{5^m\sqrt{5}}\sum_{j=0}^{2m+1}(-1)^{j-1}\binom{2m+1}{j}\sum_{k=1}^{n}\alpha^{2k(2j-2m-1)}\\
&= \frac{1}{5^m\sqrt{5}}\sum_{j=0}^{2m+1}(-1)^{j-1}\binom{2m+1}{j}\frac{\alpha^{2(n+1)(2j-2m-1)}-\alpha^{2(2j-2m-1)}}{\alpha^{2(2j-2m-1)}-1}\\
&= \frac{1}{5^m\sqrt{5}}\sum_{j=0}^{m}(-1)^{j-1}\binom{2m+1}{j}\Bigg[\frac{\alpha^{2(n+1)(2j-2m-1)}-\alpha^{2(2j-2m-1)}}{\alpha^{2(2j-2m-1)}-1}\\
&\qquad\qquad -\frac{\alpha^{2(n+1)(2m-2j+1)}-\alpha^{2(2m-2j+1)}}{\alpha^{2(2m-2j+1)}-1}\Bigg]\\
&= \frac{1}{5^m\sqrt{5}}\sum_{j=0}^{m}(-1)^{j-1}\binom{2m+1}{j}\frac{1-\alpha^{2n(2j-2m-1)}-\alpha^{2(n+1)(2m-2j+1)}+\alpha^{2(2m-2j+1)}}{\alpha^{2(2m-2j+1)}-1}\\
&= \frac{1}{5^m\sqrt{5}}\sum_{j=0}^{m}(-1)^{j-1}\binom{2m+1}{j}\frac{-\beta^{(2m-2j+1)}+\beta^{(2n+1)(2m-2j+1)}-\alpha^{(2n+1)(2m-2j+1)}+\alpha^{(2m-2j+1)}}{\alpha^{(2m-2j+1)}+\beta^{(2m-2j+1)}}\\
&= \frac{1}{5^m}\sum_{j=0}^{m}(-1)^{j}\binom{2m+1}{j}\frac{F_{(2n+1)(2m-2j+1)}-F_{2m-2j+1}}{L_{2m-2j+1}}\\
&= \frac{1}{5^m}\sum_{j=0}^{m}(-1)^{j}\binom{2m+1}{j}\frac{F_{(2n+1)(2m-2j+1)}}{L_{2m-2j+1}}-\frac{1}{5^m}\sum_{j=0}^{m}(-1)^{j}\binom{2m+1}{j}\frac{F_{2m-2j+1}}{L_{2m-2j+1}}.
\end{aligned}$$

This displays already the constant term C_m.

Now we use the lemma with $x=\alpha^{2n+1}$, and get the formula

$$F_{(2k+1)(2n+1)} = \sum_{l=0}^{k}5^l(-1)^{k-l}\frac{2k+1}{2l+1}\binom{k+l}{k-l}F_{2n+1}^{2l+1}.$$

So we can rewrite our formula:

$$\begin{aligned}
\sum_{k=1}^{n} F_{2k}^{2m+1} &= \frac{1}{5^m}\sum_{j=0}^{m}(-1)^{j}\binom{2m+1}{j}\frac{F_{(2n+1)(2m-2j+1)}}{L_{2m-2j+1}}-\frac{1}{5^m}\sum_{j=0}^{m}(-1)^{j}\binom{2m+1}{j}\frac{F_{2m-2j+1}}{L_{2m-2j+1}}\\
&= \frac{1}{5^m}\sum_{j=0}^{m}(-1)^{j}\binom{2m+1}{j}\frac{1}{L_{2m-2j+1}}\sum_{l=0}^{m-j}5^l(-1)^{m-j-l}\frac{2m-2j+1}{2l+1}\binom{m-j+l}{m-j-l}F_{2n+1}^{2l+1}
\end{aligned}$$

$$-\frac{1}{5^m}\sum_{j=0}^{m}(-1)^j\binom{2m+1}{j}\frac{F_{2m-2j+1}}{L_{2m-2j+1}}$$

$$=\sum_{l=0}^{m}F_{2n+1}^{2l+1}5^{l-m}(-1)^{m-l}\frac{1}{2l+1}\sum_{j=0}^{m-l}\binom{2m+1}{j}\binom{m-j+l}{m-j-l}\frac{2m-2j+1}{L_{2m-2j+1}}$$

$$-\frac{1}{5^m}\sum_{j=0}^{m}(-1)^j\binom{2m+1}{j}\frac{F_{2m-2j+1}}{L_{2m-2j+1}},$$

as announced.

For the reader's convenience, here is a little list:

$$\sum_{k=1}^{n}F_{2k}^{1}=F_{2n+1}^{1}-1,$$

$$\sum_{k=1}^{n}F_{2k}^{3}=\tfrac{1}{4}F_{2n+1}^{3}-\tfrac{3}{4}F_{2n+1}^{1}+\tfrac{1}{2}$$

$$\sum_{k=1}^{n}F_{2k}^{5}=\tfrac{1}{11}F_{2n+1}^{5}-\tfrac{15}{44}F_{2n+1}^{3}+\tfrac{25}{44}F_{2n+1}^{1}-\tfrac{7}{22},$$

$$\sum_{k=1}^{n}F_{2k}^{7}=\tfrac{1}{29}F_{2n+1}^{7}-\tfrac{56}{319}F_{2n+1}^{5}+\tfrac{455}{1276}F_{2n+1}^{3}-\tfrac{553}{1276}F_{2n+1}^{1}+\tfrac{139}{638},$$

$$\sum_{k=1}^{n}F_{2k}^{9}=\tfrac{1}{76}F_{2n+1}^{9}-\tfrac{189}{2204}F_{2n+1}^{7}+\tfrac{5625}{24244}F_{2n+1}^{5}-\tfrac{4083}{12122}F_{2n+1}^{3}+\tfrac{8055}{24244}F_{2n+1}^{1}-\tfrac{1877}{12122}.$$

3. Three more summation formulæ of the Melham type

A similar approach as before produces

$$\sum_{k=0}^{n}F_{2k+1}^{2m+1}=\frac{1}{5^m}\sum_{j=0}^{m}\binom{2m+1}{j}\frac{F_{2(n+1)(2m+1-2j)}}{L_{2m+1-2j}}.$$

Now we use the identity

$$x^{2k+1}-x^{-2k-1}=\sum_{l=0}^{k}\binom{k+l}{k-l}\frac{2k+1}{2l+1}\big(x-\tfrac{1}{x}\big)^{2l+1} \tag{2}$$

and its corollary ($x=\alpha^{2(n+1)}$)

$$F_{2(n+1)(2m+1-2j)}=\sum_{l=0}^{m-j}5^l\binom{m-j+l}{m-j-l}\frac{2m-2j+1}{2l+1}F_{2(n+1)}^{2l+1}$$

to continue:

$$\begin{aligned}\sum_{k=0}^{n} F_{2k+1}^{2m+1} &= \frac{1}{5^m}\sum_{j=0}^{m}\binom{2m+1}{j}\frac{F_{2(n+1)(2m+1-2j)}}{L_{2m+1-2j}}\\ &= \sum_{j=0}^{m} 5^{l-m}\binom{2m+1}{j}\frac{1}{L_{2m+1-2j}}\sum_{l=0}^{m-j}\binom{m-j+l}{m-j-l}\frac{2m-2j+1}{2l+1}F_{2(n+1)}^{2l+1}\\ &= \sum_{l=0}^{m} F_{2(n+1)}^{2l+1}\sum_{j=0}^{m-l}\binom{2m+1}{j}\frac{5^{l-m}}{L_{2m+1-2j}}\binom{m-j+l}{m-j-l}\frac{2m-2j+1}{2l+1}.\end{aligned}$$

Now we turn to the next instance:

$$\sum_{k=0}^{n} F_{2k}^{2m} = \frac{1}{5^m}\sum_{j=0}^{m-1}\binom{2m}{j}(-1)^j\frac{F_{(2n+1)(2m-2j)}}{F_{2m-2j}} + \frac{1}{5^m}\binom{2m}{m}(-1)^m(n+\tfrac12).$$

Here, we need a formula of a slightly different type.

$$x^{2N} - x^{-2N} = \big(x+\tfrac1x\big)\sum_{l=0}^{N-1}\binom{N+l}{N-l-1}\big(x-\tfrac1x\big)^{2l+1} \tag{3}$$

The factor $x+\frac1x$ is necessary here; without it, it would be like expanding an even periodic function as a Fourier series, but using only sine functions!

The corollary is ($x=\alpha^{2n+1}$)

$$F_{(2n+1)(2m-2j)} = F_{2n+1}\sum_{l=0}^{m-j-1}\binom{m-j+l}{m-j-l-1}L_{2n+1}^{2l+1}.$$

Thus

$$\begin{aligned}\sum_{k=0}^{n} F_{2k}^{2m} &= \frac{1}{5^m}\sum_{j=0}^{m-1}\binom{2m}{j}(-1)^j\frac{F_{2n+1}}{F_{2m-2j}}\sum_{l=0}^{m-j-1}\binom{m-j+l}{m-j-l-1}L_{2n+1}^{2l+1} + \frac{1}{5^m}\binom{2m}{m}(-1)^m(n+\tfrac12)\\ &= \frac{F_{2n+1}}{5^m}\sum_{l=0}^{m-1}L_{2n+1}^{2l+1}\sum_{j=0}^{m-l-1}\binom{2m}{j}(-1)^j\frac{1}{F_{2m-2j}}\binom{m-j+l}{m-j-l-1} + \frac{1}{5^m}\binom{2m}{m}(-1)^m(n+\tfrac12).\end{aligned}$$

And now we turn to the last formula:

$$\sum_{k=0}^{n} F_{2k+1}^{2m} = \frac{1}{5^m}\sum_{j=0}^{m-1}\binom{2m}{j}\frac{F_{2(n+1)(2m-2j)}}{F_{2m-2j}} + \frac{n+1}{5^m}\binom{2m}{m}.$$

We use (3) again, with $x=\alpha^{2(n+1)}$:

$$F_{2(n+1)(2m-2j)} = L_{2(n+1)}\sum_{l=0}^{m-j-1}\binom{m-j+l}{m-j-l-1}5^l F_{2(n+1)}^{2l+1},$$

whence

$$\sum_{k=0}^{n} F_{2k+1}^{2m} = \frac{1}{5^m}\sum_{j=0}^{m-1}\binom{2m}{j}\frac{1}{F_{2m-2j}}L_{2(n+1)}\sum_{l=0}^{m-j-1}\binom{m-j+l}{m-j-l-1}5^l F_{2(n+1)}^{2l+1} + \frac{n+1}{5^m}\binom{2m}{m}$$
$$= L_{2(n+1)}\sum_{l=0}^{m-1}F_{2(n+1)}^{2l+1}\sum_{j=0}^{m-l-1}\binom{2m}{j}\frac{1}{F_{2m-2j}}\binom{m-j+l}{m-j-l-1}5^{l-m} + \frac{n+1}{5^m}\binom{2m}{m}.$$

4. Summary of results

For the reader's convenience, we collect the 4 formulæ of the Melham Fibonacci type here:

$$\sum_{k=0}^{n} F_{2k}^{2m+1} = \sum_{l=0}^{m} F_{2n+1}^{2l+1}\sum_{j=0}^{m-l}(-1)^{m-l}\frac{5^{l-m}}{L_{2m-2j+1}}\binom{2m+1}{j}\binom{m-j+l}{m-j-l}\frac{2m-2j+1}{2l+1}$$
$$- \frac{1}{5^m}\sum_{j=0}^{m}(-1)^j\binom{2m+1}{j}\frac{F_{2m-2j+1}}{L_{2m-2j+1}},$$

$$\sum_{k=0}^{n} F_{2k+1}^{2m+1} = \sum_{l=0}^{m} F_{2(n+1)}^{2l+1}\sum_{j=0}^{m-l}\frac{5^{l-m}}{L_{2m-2j+1}}\binom{2m+1}{j}\binom{m-j+l}{m-j-l}\frac{2m-2j+1}{2l+1},$$

$$\sum_{k=0}^{n} F_{2k}^{2m} = \frac{F_{2n+1}}{5^m}\sum_{l=0}^{m-1}L_{2n+1}^{2l+1}\sum_{j=0}^{m-l-1}\binom{2m}{j}\binom{m-j+l}{m-j-l-1}(-1)^j\frac{1}{F_{2m-2j}}$$
$$+ \frac{1}{5^m}\binom{2m}{m}(-1)^m(n+\tfrac{1}{2}),$$

$$\sum_{k=0}^{n} F_{2k+1}^{2m} = L_{2(n+1)}\sum_{l=0}^{m-1}F_{2(n+1)}^{2l+1}\sum_{j=0}^{m-l-1}\binom{2m}{j}\binom{m-j+l}{m-j-l-1}\frac{1}{F_{2m-2j}}5^{l-m}$$
$$+ \frac{n+1}{5^m}\binom{2m}{m}.$$

5. Lucas types of Melham's sum

We start with

$$\sum_{k=0}^{n} L_{2k}^{2m+1} = \sum_{j=0}^{m}\binom{2m+1}{j}\frac{L_{(2n+1)(2m+1-2j)}}{L_{2m+1-2j}} + 4^m.$$

We use (2) with $x = \alpha^{2n+1}$:

$$L_{(2n+1)(2m+1-2j)} = \sum_{l=0}^{m-j}\binom{m-j+l}{m-j-l}\frac{2m+1-2j}{2l+1}L_{2n+1}^{2l+1}.$$

Therefore

$$\sum_{k=0}^{n} L_{2k}^{2m+1} = \sum_{j=0}^{m} \binom{2m+1}{j} \frac{1}{L_{2m+1-2j}} \sum_{l=0}^{m-j} \binom{m-j+l}{m-j-l} \frac{2m+1-2j}{2l+1} L_{2n+1}^{2l+1} + 4^m$$

$$= \sum_{l=0}^{m} L_{2n+1}^{2l+1} \sum_{j=0}^{m-l} \binom{2m+1}{j} \binom{m-j+l}{m-j-l} \frac{2m+1-2j}{2l+1} \frac{1}{L_{2m+1-2j}} + 4^m.$$

The next one:

$$\sum_{k=0}^{n} L_{2k+1}^{2m+1} = \sum_{j=0}^{m} \binom{2m+1}{j} (-1)^j \frac{L_{2(n+1)(2m+1-2j)}}{L_{2m+1-2j}} - \sum_{j=0}^{m} \binom{2m+1}{j} (-1)^j \frac{2}{L_{2m+1-2j}}.$$

We use (1) with $x = \alpha^{2(n+1)}$ to get

$$L_{2(n+1)(2m+1-2j)} = \sum_{l=0}^{m-j} (-1)^{m-j-l} \frac{2m+1-2j}{2l+1} \binom{m-j+l}{m-j-l} L_{2(n+1)}^{2l+1}.$$

Hence

$$\sum_{k=0}^{n} L_{2k+1}^{2m+1} = \sum_{j=0}^{m} \binom{2m+1}{j} (-1)^j \frac{1}{L_{2m+1-2j}} \sum_{l=0}^{m-j} (-1)^{m-j-l} \frac{2m+1-2j}{2l+1} \binom{m-j+l}{m-j-l} L_{2(n+1)}^{2l+1}$$

$$- \sum_{j=0}^{m} \binom{2m+1}{j} (-1)^j \frac{2}{L_{2m+1-2j}}$$

$$= \sum_{l=0}^{m} L_{2(n+1)}^{2l+1} \sum_{j=0}^{m-l} \binom{2m+1}{j} \binom{m-j+l}{m-j-l} \frac{2m+1-2j}{2l+1} \frac{(-1)^{m-l}}{L_{2m+1-2j}}$$

$$- \sum_{j=0}^{m} \binom{2m+1}{j} (-1)^j \frac{2}{L_{2m+1-2j}}.$$

Furthermore,

$$\sum_{k=0}^{n} L_{2k}^{2m} = \sum_{j=0}^{m-1} \binom{2m}{j} \frac{F_{(2n+1)(2m-2j)}}{F_{2m-2j}} + 2^{2m-1} + \binom{2m}{m} \big(n + \tfrac{1}{2}\big).$$

Now we need the formula (3) and its corollary for $x = \alpha^{2n+1}$:

$$F_{(2n+1)(2m-2j)} = F_{2n+1} \sum_{l=0}^{m-j-1} \binom{m-j+l}{m-j-l-1} L_{2n+1}^{2l+1}.$$

Hence

$$\sum_{k=0}^{n} L_{2k}^{2m} = \sum_{j=0}^{m-1} \binom{2m}{j} \frac{F_{2n+1}}{F_{2m-2j}} \sum_{l=0}^{m-j-1} \binom{m-j+l}{m-j-l-1} L_{2n+1}^{2l+1} + 2^{2m-1} + \binom{2m}{m} \big(n + \tfrac{1}{2}\big)$$

$$= F_{2n+1}\sum_{l=0}^{m-1} L_{2n+1}^{2l+1}\sum_{j=0}^{m-1-l}\binom{2m}{j}\binom{m-j+l}{m-j-l-1}\frac{1}{F_{2m-2j}} + 2^{2m-1} + \binom{2m}{m}\big(n+\tfrac12\big).$$

Finally

$$\sum_{k=0}^{n} L_{2k+1}^{2m} = \sum_{j=0}^{m-1}\binom{2m}{j}(-1)^j\frac{F_{2(n+1)(2m-2j)}}{F_{2m-2j}} + \binom{2m}{m}(-1)^m(n+1).$$

We use (2) again to get

$$F_{2(n+1)(2m-2j)} = L_{2(n+1)}\sum_{l=0}^{m-j-1}\binom{m-j+l}{m-j-l-1}5^l F_{2(n+1)}^{2l+1}.$$

Therefore

$$\sum_{k=0}^{n} L_{2k+1}^{2m} = \sum_{j=0}^{m-1}\binom{2m}{j}(-1)^j\frac{1}{F_{2m-2j}}L_{2(n+1)}\sum_{l=0}^{m-j-1}\binom{m-j+l}{m-j-l-1}5^l F_{2(n+1)}^{2l+1} + \binom{2m}{m}(-1)^m(n+1)$$

$$= L_{2(n+1)}\sum_{l=0}^{m-1} F_{2(n+1)}^{2l+1}\sum_{j=0}^{m-1-l}(-1)^j\binom{2m}{j}\binom{m-j+l}{m-j-l-1}\frac{5^l}{F_{2m-2j}} + \binom{2m}{m}(-1)^m(n+1).$$

6. Collection of the Lucas Melham formulæ

$$\sum_{k=0}^{n} L_{2k}^{2m+1} = \sum_{l=0}^{m} L_{2n+1}^{2l+1}\sum_{j=0}^{m-l}\binom{2m+1}{j}\binom{m-j+l}{m-j-l}\frac{2m+1-2j}{2l+1}\frac{1}{L_{2m+1-2j}} + 4^m,$$

$$\sum_{k=0}^{n} L_{2k+1}^{2m+1} = \sum_{l=0}^{m} L_{2(n+1)}^{2l+1}\sum_{j=0}^{m-l}\binom{2m+1}{j}\binom{m-j+l}{m-j-l}\frac{2m+1-2j}{2l+1}\frac{(-1)^{m-l}}{L_{2m+1-2j}} - \sum_{j=0}^{m}\binom{2m+1}{j}(-1)^j\frac{2}{L_{2m+1-2j}}$$

$$\sum_{k=0}^{n} L_{2k}^{2m} = F_{2n+1}\sum_{l=0}^{m-1} L_{2n+1}^{2l+1}\sum_{j=0}^{m-1-l}\binom{2m}{j}\binom{m-j+l}{m-j-l-1}\frac{1}{F_{2m-2j}}$$

$$+ 2^{2m-1} + \binom{2m}{m}\big(n+\tfrac{1}{2}\big),$$

$$\sum_{k=0}^{n} L_{2k+1}^{2m} = L_{2(n+1)} \sum_{l=0}^{m-1} F_{2(n+1)}^{2l+1} \sum_{j=0}^{m-1-l} (-1)^j \binom{2m}{j} \binom{m-j+l}{m-j-l-1} \frac{5^l}{F_{2m-2j}} + \binom{2m}{m} (-1)^m (n+1).$$

7. Applications

With these results, we can iterate the summations, for instance we can sum the Melham sum, viz.

$$\begin{aligned}\sum_{n=0}^{N}\sum_{k=0}^{n} F_{2k}^{2m+1} &= \sum_{n=0}^{N}\sum_{l=0}^{m} F_{2n+1}^{2l+1} 5^{l-m} (-1)^{m-l} \frac{1}{2l+1} \sum_{j=0}^{m-l} \binom{2m+1}{j}\binom{m-j+l}{m-j-l} \frac{2m-2j+1}{L_{2m-2j+1}}\\ &\quad - \sum_{n=0}^{N} \frac{1}{5^m} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{j} \frac{F_{2m-2j+1}}{L_{2m-2j+1}}\\ &= \sum_{l=0}^{m} 5^{l-m} (-1)^{m-l} \frac{1}{2l+1} \sum_{j=0}^{m-l} \binom{2m+1}{j}\binom{m-j+l}{m-j-l} \frac{2m-2j+1}{L_{2m-2j+1}} \times\\ &\quad \times \sum_{a=0}^{l} F_{2(N+1)}^{2a+1} \sum_{b=0}^{l-a} \frac{5^{a-l}}{L_{2l-2b+1}} \binom{2l+1}{b}\binom{l-b+a}{l-b-a} \frac{2l-2b+1}{2a+1}\\ &\quad - (N+1)\frac{1}{5^m} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{j} \frac{F_{2m-2j+1}}{L_{2m-2j+1}}.\end{aligned}$$

The coefficients of $F_{2(N+1)}^{2a+1}$ are now triple sums. In principle, further iterations can be performed, but the results are not too attractive.

References

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