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TOPOLOGIES ON FREE MONOIDS INDUCED BY FAMILIES OF LANGUAGES (*)

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Abstract. — *For* $\mathscr{L} \subseteq \mathscr{P}(\Sigma^*)$ *the language operator* $\mathrm{Anf}_{\mathscr{L}}(A)$ *is defined by* $\{ z \mid z \diagdown A \in \mathscr{L} \}$. *It was characterized what families* $\mathscr{L}$ *correspond to closure operators. In this paper the families* $\mathscr{L}$ *are found out corresponding to interior operators: they are filters with a special property. For the case of principal filters* $\mathscr{L} = \{ A \mid A \supseteq L \}$ *such a family is obtained iff L is a monoid. Thus from every monoid a topology can be constructed. Further results are given.*

Résumé. — *Étant donné une classe de langages* $\mathscr{L}$, *on définit un opérateur sur les langages* $\mathrm{Anf}_{\mathscr{L}}(A) = \{ z \mid z \diagdown A \in \mathscr{L} \}$. *On connaissait déjà les familles* $\mathscr{L}$ *correspondant à des opérateurs de fermeture. Dans cet article on décrit les familles* $\mathscr{L}$ *correspondant à des opérateurs d'ouverture : ce sont des filtres avec une propriété caractéristique. Pour le cas de filtres principaux* $\mathscr{L} = \{ A \mid A \supseteq L \}$ *cette propriété caractéristique est que L soit un monoïde. Par conséquent on peut construire une topologie pour chaque monoïde L. D'autres résultats sont formulés dans l'article.*

1. INTRODUCTION

In [2] there are considered some special topologies on the free monoid Σ^*. For the sake of brevity, the reader is assumed to have a certain knowledge of this paper. If $\mathscr{L}$ is a family of languages, let $\mathrm{Anf}_{\mathscr{L}}(A) = \{ z \mid z \diagdown A \in \mathscr{L} \}$. It has been characterized in terms of 4 axioms what families $\mathscr{L}$ produce *closure operators* $\mathrm{Anf}_{\mathscr{L}}$. (So we know what families induce a topology on Σ^*; from now on we call them $\mathscr{L}$*-topologies.*) Furthermore it was possible to know from the *family of open sets* whether or not the topology on Σ^* was an $\mathscr{L}$-topology.

In Section 2 we make some further remarks on our former paper.

It is well known that a topology can be described in some ways: closure operator, family of open sets, *interior operator*, *neighbourhood system*, etc. (We refer for topological conceptions to [1].) The first two ways with respect to $\mathscr{L}$-topologies are already considered in [2]; in Sections 3 and 4 the third and fourth possibility of generating an $\mathscr{L}$-topology are discussed.

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2. ADDITIONAL REMARKS ON OUR FIRST STUDY OF $\mathscr{L}$-TOPOLOGIES

We present a further example of an $\mathscr{L}$-topology: Let $A/w=\{z \mid zw\in A\}$ and assume $z\in\Sigma^*$ to be fixed. Let $\varphi_z(A):=\bigcup_{n\geqq 0} A/z^n$. It is easy to see that φ_z fulfills the axioms (A1)-(A4) and is therefore a closure operator. Now, since $(x\diagdown A)/y=x\diagdown(A/y)$, it follows that:

$$\varphi_z(w\diagdown A)=w\diagdown\varphi_z(A) \quad \text{for all } w\in\Sigma^*.$$

So φ_z is *leftquotient-permutable* and thus by Lemma 2.7 of [2] $\varphi_z=\mathrm{Anf}_{\mathscr{L}_z}$, where $\mathscr{L}_z=\{A \mid \varepsilon\in\varphi_z(A)\}=\{A \mid \text{there exists an } n\in N_0 \text{ such that } z^n\in A\}$. For $z=\varepsilon$ we obtain the *discrete topology*.

It is clear how this situation can be generalized. Let $M\subseteqq\Sigma^*$ be a *submonoid* and $\varphi_M(A):=\bigcup_{m\in M} A/m$, then φ_M is the closure operator of an $\mathscr{L}$-topology with $\mathscr{L}_M=\{A \mid M\cap A\neq\emptyset\}$.

We present in short some examples of topologies which are *not* $\mathscr{L}$-topologies:

The closure operator $L\mapsto L\Sigma^*$; the closure operator $L\mapsto\Sigma^* L$; the (*so called*) *left topology*; let us recall that the right topology *is* an $\mathscr{L}$-topology (with closure operator Init).

THEOREM 2.1: *The following* 3 *statements are equivalent:*

(i) *$X_{\mathscr{L}}$ is a T_1-space (i. e. each set $\{x\}$ is closed);*

(ii) *$\partial(\mathscr{L})$ contains no set of cardinality* 1;

(iii) *$\partial(\mathscr{L})$ contains no finite set.*

Proof: The equivalence of (i) and (ii) has been already proved in [2]. Trivially, (iii) implies (ii). Now assume that (i) holds and $L\in\partial(\mathscr{L})$ be a finite set. Then, by (i), L is closed. But a set L in $\partial(\mathscr{L})$ can never be closed, because $\varepsilon\diagdown L=L\in\partial(\mathscr{L})\subseteqq\mathscr{L}$ and $\varepsilon\notin L$.

3. INTERIOR OPERATORS AND $\mathscr{L}$-TOPOLOGIES

For a given topology, let I be the *interior operator*, defined by $I(A)=(\overline{A^c})^c$ (sometimes written as A^0).

THEOREM 3.1: *The interior operator of an $\mathscr{L}$-topology is leftquotient-permutable; the corresponding family $\mathscr{L}_I$ is given by:*

$$\mathscr{L}_I=\{A \mid A^c\notin\mathscr{L}\}.$$

Proof: Since $(x\backslash B)^c = x\backslash B^c$ and $\mathrm{Anf}_{\mathscr{L}}(x\backslash B) = x\backslash \mathrm{Anf}_{\mathscr{L}}(B)$, we have:

$$I(x\backslash A) = [\mathrm{Anf}_{\mathscr{L}}((x\backslash A)^c)]^c = [\mathrm{Anf}_{\mathscr{L}}(x\backslash A^c)]^c = [x\backslash \mathrm{Anf}_{\mathscr{L}}(A^c)]^c$$
$$= x\backslash [\mathrm{Anf}_{\mathscr{L}}(A^c)]^c = x\backslash I(A).$$

By [2]; Lemma 2.7, $\mathscr{L}_I = \{A \mid \varepsilon \in I(A)\}$. Now we have:

$$\varepsilon \in I(A) \Leftrightarrow \varepsilon \in [\mathrm{Anf}_{\mathscr{L}}(A^c)]^c$$
$$\Leftrightarrow \varepsilon \notin \mathrm{Anf}_{\mathscr{L}}(A^c) \Leftrightarrow \varepsilon\backslash A^c \notin \mathscr{L} \Leftrightarrow A^c \notin \mathscr{L},$$

thus $A \in \mathscr{L}_I \Leftrightarrow A^c \notin \mathscr{L}$.

Example: For $\mathscr{L} = \mathscr{P}_0(\Sigma^*)$, we have $\mathscr{L}_I = \{\Sigma^*\}$; $z \in I(A) \Leftrightarrow$ for all x holds $zx \in A$.

For $\mathscr{L} = \mathscr{U} \cup \{A \mid \varepsilon \in A\}$, we have $\mathscr{L}_I = \{A \mid A^c$ finite and $\varepsilon \in A\}$; $z \in I(A) \Leftrightarrow z \in A$ and for almost all x holds $zx \in A$.

In [2] there are given 4 axioms (T1)-(T4) which characterize the $\mathscr{L}$'s leading to closure operators [$\alpha(\mathscr{L}) = \mathscr{L}$ is assumed to hold].

A straightforward reformulation of this axioms in terms of $\mathscr{L}_I$ yields:

THEOREM 3.2: *Let* $\mathscr{L}_I \subseteqq \{A \mid \varepsilon \in A\}$. *Then* $\mathscr{L}_I$ *leads to an interior operator iff* (I1)-(I4) *hold:*

$$\Sigma^* \in \mathscr{L}_I, \tag{I1}$$
$$A \in \mathscr{L}_I,\quad A \subseteqq B \Rightarrow B \in \mathscr{L}_I, \tag{I2}$$
$$A \in \mathscr{L}_I,\quad B \in \mathscr{L}_I \Rightarrow A \cap B \in \mathscr{L}_I, \tag{I3}$$
$$A \in \mathscr{L}_I \Leftrightarrow \mathrm{Anf}_{\mathscr{L}_I}(A) \in \mathscr{L}_I, \tag{I4}$$

REMARK: Similar as for $\mathscr{L}$ in [2], it is possible to drop the condition $\mathscr{L}_I \subseteqq \{A \mid \varepsilon \in A\}$ and to formulate other axioms. But this is not too meaningful and therefore omitted.

REMARK: Since $\Sigma^* \in \mathscr{L}$, it follows $\emptyset \notin \mathscr{L}_I$. This together with (I1)-(I3) leads to the surprising fact that:

$\mathscr{L}_I$ is a (proper) filter.

So the question arise what filters fulfill the axiom (I4). For the special case of a *principal filter* $\mathscr{L}(L) := \{A \mid A \supseteqq L\}$ this can be answered:

THEOREM 3.3: $\mathscr{L}(L)$ *fulfills axiom* (I4) *iff* $\mathscr{L}$ *is a monoid.*

Proof: Let us reformulate axiom (I4) for this special situation: $A \in \mathscr{L}(L) \Leftrightarrow \mathrm{Anf}_{\mathscr{L}(L)}(A) \in \mathscr{L}(L)$ means:

$$L \subseteqq A \Leftrightarrow L \subseteqq \mathrm{Anf}_{\mathscr{L}(L)}(A) \Leftrightarrow L \subseteqq \{z \mid L \subseteqq z\backslash A\}.$$

Thus axiom (I4) is equivalent to:

$$L \subseteq A \quad \Leftrightarrow \quad [z \in L \Rightarrow L \subseteq z \diagdown A]. \tag{$*$}$$

Setting $A = L$, $(*)$ implies:

$$z \in L \quad \Rightarrow \quad L \subseteq z \diagdown L. \tag{$**$}$$

But a short reflection shows that $(**)$ is also equivalent to $(*)$ [and to (I4)!] Furthermore this means:

$$z \in L \quad \Rightarrow \quad [w \in L \Rightarrow w \in z \diagdown L],$$

or:

$$z \in L, \quad w \in L \quad \Rightarrow \quad zw \in L.$$

Since $\mathscr{L}(L) \subseteq \{A \mid \varepsilon \in A\}$ we have $\varepsilon \in L$, and the proof is finished.

REMARK: Each submonoid $M \subseteq \Sigma^*$ leads us to an $\mathscr{L}$-topology!

Let us recall the following fact from [2]: Let $X = (\Sigma^*, \mathfrak{O})$ be an $\mathscr{L}$-topology. Then:

$$\mathscr{L} = \mathscr{P}(\Sigma^*) - \{A \mid \text{there is an } 0 \in \mathfrak{O} \text{ such that } \varepsilon \in 0 \text{ and } A \subseteq 0^c\};$$

this family $\mathscr{L}$ is unique subject to the condition $\mathscr{L} = \alpha(\mathscr{L})$. Now let us compute $\mathscr{L}_I$:

$$A \in \mathscr{L}_I \quad \Leftrightarrow \quad A^c \notin \mathscr{L} \quad \Leftrightarrow \quad A^c \in \{B \mid \exists 0 \in \mathfrak{O} : \varepsilon \in 0 \text{ and } B \subseteq 0^c\}$$

$$\Leftrightarrow \quad \exists 0 \in \mathfrak{O} : \varepsilon \in 0 \quad \text{and} \quad A^c \subseteq 0^c \quad \Leftrightarrow \quad \exists 0 \in \mathfrak{O} : \varepsilon \in 0 \subseteq A;$$

$$\mathscr{L}_I = \{A \mid \exists 0 \in \mathfrak{O} : \varepsilon \in 0 \subseteq A\}$$

and we find:

$\mathscr{L}_I$ is the filter of neighbourhoods of ε!

By [2]; Lemma 2.13, we know A open $\Leftrightarrow$ for all $x \in A$ holds $(x \diagdown A)^c \notin \mathscr{L}$, which now simply means:

for all $x \in A$ holds $x \diagdown A \in \mathscr{L}_I$!

Altogether it seems that it is easier to work with $\mathscr{L}_I$ instead of $\mathscr{L}$!

Now we are ready to formulate a *general base representation theorem* (generalizing [2]; Theorems 3.3 and 3.4):

THEOREM 3.4: *Let $X = (\Sigma^*, \mathfrak{O})$ be an $\mathscr{L}$-topology. Then:*

$$\mathfrak{B} = \{xA \mid x \in \Sigma^*, A \in \mathscr{L}_I\} \text{ is a base for } \mathfrak{O}.$$

Proof: If 0 is open, then for all $x \in 0$ holds $x \backslash 0 \in \mathscr{L}_I$. Thus $x(x \backslash 0) \in \mathfrak{B}$ and $0 = \bigcup_{x \in 0} x(x \backslash 0)$.

4. SYSTEMS OF NEIGHBOURHOODS AND $\mathscr{L}$-TOPOLOGIES

A further method to generate a topology is to construct a system of neighbourhoods.

THEOREM 4.1: *Let $X = (\Sigma^*, \mathfrak{O})$ be an $\mathscr{L}$-topology and let $\mathfrak{B}(x)$ be the family of neighbourhoods of x. Then:*

$$\mathfrak{B}(x) = y \backslash \mathfrak{B}(yx).$$

Proof:

$$\mathfrak{B}(x) = \{ A \mid \exists 0 \in \mathfrak{O} : x \in 0 \subseteqq A \} = \{ A \mid \exists 0 \in \mathfrak{O} : \varepsilon \in x \backslash 0 \subseteqq x \backslash A \};$$

$$\begin{aligned} y \backslash \mathfrak{B}(yx) &= y \backslash \{ A \mid \exists 0 \in \mathfrak{O} : \varepsilon \in yx \backslash 0 \subseteqq yx \backslash A \} \\ &= y \backslash \{ A \mid \exists 0 \in \mathfrak{O} : \varepsilon \in x \backslash (y \backslash 0) \subseteqq x \backslash (y \backslash A) \} \\ &= \{ y \backslash A \mid \exists 0 \in \mathfrak{O} : \varepsilon \in x \backslash (y \backslash 0) \subseteqq x \backslash (y \backslash A) \} \\ &= \{ A \mid \exists 0 \in \mathfrak{O} : \varepsilon \in x \backslash 0 \subseteqq x \backslash A \}. \end{aligned}$$

REMARK: The property $\mathfrak{B}(x) = y \backslash \mathfrak{B}(yx)$ implies $y \mathfrak{B}(x) \subseteqq \mathfrak{B}(yx)$.

We can prove also a converse of Theorem 4.1.

THEOREM 4.2: *Assume that there is a system of neighbourhoods $\{ \mathfrak{B}(x) \}$ satisfying:*

$$\mathfrak{B}(x) = y \backslash \mathfrak{B}(yx).$$

Then the topology is even an $\mathscr{L}$-topology.

Proof: By [2]; Theorem 2.16 it is sufficient to show that the system of open sets $\mathfrak{O}$ is *left stable.*

Let 0 be open, i. e. 0 is neighbourhood of all its points, i. e.:

$$x \in 0 \quad \Rightarrow \quad 0 \in \mathfrak{B}(x).$$

To show: $z \backslash 0$ is open. Let $x \in z \backslash 0$, i. e. $zx \in 0$, i. e. $0 \in \mathfrak{B}(zx)$. By the condition: $z \backslash 0 \in z \backslash \mathfrak{B}(zx) = \mathfrak{B}(x)$.

Furthermore we have to show: $z0$ is open. Let $x \in z0$, i. e. $x = zy$ and $y \in 0$, i. e. $0 \in \mathfrak{B}(y)$. From the last remark: $z0 \in z\mathfrak{B}(y) \subseteqq \mathfrak{B}(zy) = \mathfrak{B}(x)$.

REMARK: We know already that $\mathscr{L}_I$ is simply $\mathfrak{B}(\varepsilon)$. So we have for all systems of neighbourhoods by means of the remark after Theorem 4.1:

$$\mathfrak{B}(x) \supseteqq x\,\mathfrak{B}(\varepsilon) = x\,\mathscr{L}_I.$$

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