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AN ELEMENTARY APPROACH TO THE STACK SIZE OF REGULARLY DISTRIBUTED BINARY TREES

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For the sake of brevity we assume that the reader has a certain knowledge of [2]. Let T be a *binary tree* with n leaves. Evaluating T in postorder it is assumed that in one unit of time a node is stored in the stack or is removed from the top of the stack. Consider the number of nodes stored in the stack after t units of time. Let $R_d(n, t)$ denote the d-th moment with respect to the origin of this statistic. In [1] R. Kemp was able to produce an *exact formula* for $R_1(n, t)$ by use of 2 combinatorial identities. These identities are generalized and more easily proved in [3]. As stated in [2], a similar approach would give *exact formulas for* $R_d(n,t)$, d odd, *but not for even d.* For that purpose R. Kemp used a complex variable approach to give *asymptotic equivalents* for the numbers $R_d(n, t)$, assuming that $n, t \to \infty$ and $t \to 2\rho n$, $0 < \rho < 1$, ρ a constant. He obtains

$$R_d(n, t) = \pi^{-1/2}\, 2^{d+1}\, \Gamma\left(\frac{d+3}{2}\right) \cdot (n\rho(1-\rho))^{d/2} + O(n^{(d-1)/2})\,. \tag{1}$$

Here we show that it is possible to give exact formulas for $R_d(n, t)$ *for all d* by *elementary methods.* For instance, we give a formula for $R_2(n, t)$ from which an exact formula for the variance can be obtained by

$$\sigma^2(n, t) = R_2(n, t) - R_1^2(n, t)$$

and Kemp's formula for $R_1(n, t)$ (see [1]).

$$R_1(n, 2t) = \frac{2t(n-t)(2n-1)}{(n-1)\,n} \cdot \varphi(n, t)$$

$$(2) \qquad R_1(n, 2t+1) = \frac{(2t+1)(2n-2t-1)-n}{n-1} \cdot \frac{2n-1}{2n-2t-1} \cdot \frac{n-t}{n} \cdot \varphi(n, t)$$

with

$$\varphi(n, t) = \binom{2t}{t}\binom{2n-2t}{n-t}\binom{2n}{n}^{-1}.$$

It is known [2] that

$$(3) \qquad R_d(n, 2t+s) = \frac{2(2n-1)}{(2t+s)(2n-2t-s)}\binom{2n}{n}^{-1} \cdot \sum_{k\geq 0} (2k+s)^{d+2}\binom{2t+s}{t-k}\binom{2n-2t-s}{n-t-s-k}.$$

Let

$$(4) \qquad f(d, s; m, n) := \sum_{k\geq 1} (2k+s)^d \binom{2m+s}{m-k}\binom{2n+s}{n-k},$$

$d, s, m, n \in N_0$. We propose to show how a closed formula for $f(d, s; m, n)$ can be obtained which is obviously equivalent to the same problem for $R_d(n, t)$. The method is essentially included in [3].

Theorem 1. The following recursion holds for the numbers $f(d, s; m, n)$:

$$(5) \qquad f(d+2, s; m, n) = (2m+s)^2 f(d, s; m, n) - 4(2m+s)_2 f(d, s; m-1, n).$$

Here, $(x)_k$ denotes the *falling factorials*.

Proof. Since

$$(6) \qquad \binom{2m-2+s}{m-1-k} = \frac{(m-k)(m+k+s)}{(2m+s)_2}\binom{2m+s}{m-k}$$

and

$$(7) \qquad 4(m-k)(m+k+s) = (2m+s)^2 - (2k+s)^2,$$

a rearrangement of (6) and summation over $k \geq 1$ gives (5). □

So if we have formulas for $d = 0$ and 1, we have solved our problem. $f(1, s; m, n)$ is known [3]:

$$f(1, s; m, n) = \binom{2m+s}{m}\binom{2n+s}{n}\frac{mn}{m+n+s}. \tag{8}$$

Theorem 2.

$$\begin{aligned} f(0, s; m, n) \\ = \frac{1}{2}\left[\binom{2m+2n+2s}{m+n+s} - \sum_{0\le k\le s}\binom{2m+s}{m+k}\binom{2n+s}{n+k}\right]. \end{aligned} \tag{9}$$

Proof. Let without loss of generality $m \le n$.

$$\begin{aligned} \xi = & \sum_{0\le k\le 2m+s}\binom{2m+s}{k}\binom{2n+s}{m+n+s-k} + \binom{2m+s}{m}\binom{2n+s}{n} \\ & - \sum_{m\le k\le 2m+s}\binom{2m+s}{k}\binom{2n+s}{n-m+k} \\ = & \binom{2m+2n+2s}{m+n+s} + \binom{2m+s}{m}\binom{2n+s}{n} \\ & - \xi - \sum_{1\le k\le s}\binom{2m+s}{m+k}\binom{2n+s}{n+k}, \end{aligned}$$

which gives (9), since

$$\begin{aligned} \xi & = \sum_{0\le k\le m}\binom{2m+s}{m-k}\binom{2n+s}{n-k} \\ & = f(0, s; m, n) + \binom{2m+s}{m}\binom{2n+s}{n}. \quad \square \end{aligned}$$

Obviously, (9) is only useful for small values of s. However, in practice we require just $s = 0$ and $s = 1$. By doing some computations using Theorem 1 we obtain

Corollary 3.

$$f(2, 0; m, n) = \binom{2m+2n}{m+n}\frac{2mn}{2m+2n-1}, \tag{10}$$

$$f(4, 0; m, n) = \binom{2m+2n}{m+n}\frac{8mn(3mn-m-n)}{(2m+2n-1)(2m+2n-3)} \tag{11}$$

$$\begin{aligned} & f(2, 1; m, n) \\ & = \frac{1}{2}\binom{2m+2n+2}{m+n+1}\frac{2m+2n+1+4mn}{2m+2n+1} \\ & \quad - \binom{2m+1}{m}\binom{2n+1}{n} \\ & = f(0, 1; m, n) + \frac{2mn}{2m+2n+1}\binom{2m+2n+2}{m+n+1}, \end{aligned} \tag{12}$$

$$
\begin{aligned}
&f(4,\,1;\;m,\,n)\\
&\quad=\frac{1}{2}\binom{2m+2n+2}{m+n+1}\frac{(2m+1)(2n+1)(12mn+2m+2n-1)}{(2m+2n+1)(2m+2n-1)}\\
&\qquad-\binom{2m+1}{m}\binom{2n+1}{n}\\
&\quad=f(2,\,1;\;m,\,n)\\
&\qquad+\binom{2m+2n+2}{m+n+1}\frac{6mn(2m+1)(2n+1)}{(2m+2n+1)(2m+2n-1)}\,. \quad\square
\end{aligned}
\tag{13}
$$

Since

$$
\begin{aligned}
&R_d(n,\,2t+s)\\
&\quad=\frac{2(2n-1)}{(2t+s)(2n-2t-s)}\binom{2n}{n}^{-1}\\
&\qquad\cdot\left[s^{d+2}\binom{2t+s}{t}\binom{2n-2t-s}{n-t}\right.\\
&\qquad\left.+f(d+2,\,s;\;t,\,n-t-s)\right],
\end{aligned}
\tag{14}
$$

an obvious computation gives

Theorem 4.

$$
\begin{aligned}
R_2(n,\;2t)&=\frac{12t(n-t)-4n}{2n-3},\\
R_2(n,\;2t+1)&=\frac{12t(n-t-1)+2n-3}{2n-3}\,. \quad\square
\end{aligned}
\tag{15}
$$

It is trivial to obtain an asymptotic formula for $R_2(n,\,t)$, $n\to\infty$, $t\to\rho n$.

It is worthwile to discuss the limitations of our elementary approach: A general asymptotic formula like (1) *cannot* be obtained, although the recursion (5) seems to be promising. But unfortunately the two terms of the right-hand side of (5) are of equal rate of growth.

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