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On the moments of a distribution defined by the Gaussian polynomials

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Abstract

An alternative method is presented to compute the moments of the probability distribution defined by the Gaussian polynomials. It computes the cumulants first.

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Di Bucchianico (1999) has considered the probability distribution defined by the (probability) generating function

$$F_{m,n}(q) = \frac{\begin{bmatrix} m+n \\ m \end{bmatrix}_q}{\binom{m+n}{m}},$$

with the Gaussian polynomials

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \frac{(1-q)(1-q^2)\cdots(1-q^{m+n})}{(1-q)(1-q^2)\cdots(1-q^m)(1-q)(1-q^2)\cdots(1-q^n)}.$$

He discussed several methods to compute the moments of this distribution.

In this short note, I want to draw the attention of the reader to another (potentially superior) method that is due to Panny (1986). This method first computes the *cumulants* and translates them into the *moments* by a standard formula. This yields *explicit* formulæ for all the moments.

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Note that

$$F_{m,n}(q)=\frac{g_1(q)\cdots g_{m+n}(q)}{g_1(q)\cdots g_m(q)g_1(q)\cdots g_n(q)}$$

with

$$g_k(q)=\frac{1-q^k}{k(1-q)},$$

where $g_k(q)$ is a probability generating function of a random variable X_k. The function $\log g_k(\mathrm{e}^{\mathrm{i}t})$ is the generating function of the cumulants.[1] Panny has computed that as

$$\log g_k(\mathrm{e}^{\mathrm{i}t})=\frac{(k-1)\mathrm{i}t}{2}+\sum_{j\geqslant 1}\frac{B_{2j}}{2j}(k^{2j}-1)\frac{(\mathrm{i}t)^{2j}}{(2j)!},\qquad |t|<\frac{2\pi}{k},$$

where B_i denotes the ith Bernoulli number.

Reading off coefficients we find the cumulants of X_k as

$$\kappa_1=\frac{k-1}{2},\qquad \kappa_{2r}=\frac{(k^{2r}-1)B_{2r}}{2r},\ \ \kappa_{2r+1}=0,\ \ r=1,2,3,\ldots\ .$$

Since

$$\log F_{m,n}(\mathrm{e}^{\mathrm{i}t})=\sum_{k=1}^{m+n}\log g_k(\mathrm{e}^{\mathrm{i}t})-\sum_{k=1}^{m}\log g_k(\mathrm{e}^{\mathrm{i}t})-\sum_{k=1}^{n}\log g_k(\mathrm{e}^{\mathrm{i}t}),$$

the cumulants $\kappa_r^{m,n}$ are obtained by summing the κ_r's.

So we get

$$\kappa_1^{m,n}=\frac{\binom{m+n}{2}}{2}-\frac{\binom{m}{2}}{2}-\frac{\binom{n}{2}}{2}=\frac{mn}{2},$$

$$\kappa_{2r}^{m,n}=\frac{B_{2r}}{2r(2r+1)}\left(B_{2r+1}(m+n+1)-B_{2r+1}(m+1)-B_{2r+1}(n+1)\right),$$

$$\kappa_{2r+1}^{m,n}=0.$$

Here is a little list:

$$\kappa_1^{m,n}=\frac{mn}{2},$$

$$\kappa_2^{m,n}=\frac{mn(m+n+1)}{12},$$

$$\kappa_4^{m,n}=-\frac{mn(m+n+1)(m(m+1)+mn+n(n+1))}{120}.$$

[1] We write $\log x$ for the (natural) logarithm with base e.

The formula to compute the moments μ_r from this is

$$\mu_r = \sum_{\pi_1+2\pi_2+\cdots+r\pi_r=r} \left(\frac{\kappa_1}{1!}\right)^{\pi_1} \left(\frac{\kappa_2}{2!}\right)^{\pi_2} \cdots \left(\frac{\kappa_r}{r!}\right)^{\pi_r} \frac{r!}{\pi_1!\pi_2!\ldots\pi_r!},$$

which is a finite sum since $\pi_k \in \{0, 1, \ldots, r\}$. Hence

$$\mu_1 = \frac{mn}{2},$$

$$\mu_2 = \kappa_1^2 + \kappa_2 = \frac{mn(m+n+3mn+1)}{12},$$

$$\mu_3 = \kappa_1^3 + 3\kappa_1\kappa_2 + \kappa_3 = \frac{m^2(m+1)n^2(n+1)}{8},$$

$$\vdots$$

We finish by mentioning that, as in (Panny, 1986), one could also get asymptotic formulæ.

References

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