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On the Recursion Depth of Special Tree Traversal Algorithms*

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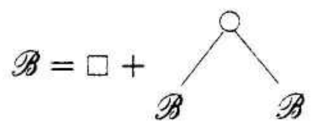
The performance of several tree traversal algorithms may be described in terms of various notions of height. Some results on the statistics of these parameters are obtained by means of generating function techniques.

1. INTRODUCTION

In this paper we are concerned with the analysis of special recursive algorithms for traversing the nodes of a planted plane tree (ordered tree; planar tree). Some by now classical results in this area are due to (de Bruijn, Knuth, and Rice, 1972; Knuth, 1973; Flajolet, 1981; Flajolet and Odlyzko, 1982; Flajolet, Raoult, and Vuillemin, 1979; Kemp, 1979) and others and are summarized in the next few lines:

The most important tree structure in computer science is the binary tree. The family $\mathscr{B}$ of binary trees fulfills the symbolic equation



expressing the fact that a tree in $\mathscr{B}$ may be either the empty tree □ or a tree consisting of a root ○ followed by a left and a right subtree being elements of $\mathscr{B}$ again. The inorder traversal (Knuth, 1973) is the following recursive principle:

Traverse the left subtree
Visit the root
Traverse the right subtree

The most straigthforward implementation uses an auxiliary stack to keep necessary nodes of the tree. The analysis of the expected time of the visit

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procedure is clearly linear in the size of the input tree. To evaluate recursion depth means to determine the average stack height as a function of the size (i.e., the number of internal nodes ○) of the tree. The recursion depth or height h of the binary tree is recursively determined as follows.

$$h(\square) = 0$$

and

$$h\left(\begin{array}{ccc} & \circ & \\ / & & \backslash \\ t_1 & & t_2 \end{array}\right) = 1 + \max\{h(t_1), h(t_2)\}.$$

We remark that $h(t)$ is the number of edges in the longest chain connecting the root of t with a leaf (external node □).

In (Flajolet and Odlyzko, 1982) the average value h_n of $h(t)$ in the family $\mathscr{B}_n$ of binary trees of size n is shown to be

$$h_n \sim 2\sqrt{\pi n}.$$

The recursive visit procedure can be optimized in the case of binary trees by eliminating endrecursion: the resulting iterative algorithm at each stage keeps a list of right subtrees that still remain to be explored. The storage complexity of this optimized algorithm is easily seen to correspond to the so-called left-sided height h^* defined by

$$h^*(\square) = 0,$$

$$h^*\left(\begin{array}{ccc} & \circ & \\ / & & \backslash \\ t_1 & & t_2 \end{array}\right) = \max\{1 + h^*(t_1), h^*(t_2)\}.$$

Recalling that the rotation correspondence (Knuth, 1973, 2.3.2) transforms a binary tree of $n-1$ internal (binary) nodes into a planted plane tree with n nodes, the average storage complexity of the optimized algorithm follows immediately by a result of (de Bruijn, Knuth, and Rice, 1972) about the average height of planted plane trees:

$$h_n^* \sim \sqrt{\pi n},$$

where the index n again refers to the size of the trees.

It was already proposed in Knuth's book to consider this kind of question for other families of trees. Dealing with the family $\mathscr{P}$ of planted plane trees defined by

$$\mathscr{P} = \circ + \begin{array}{c} \circ \\ | \\ \mathscr{P} \end{array} + \begin{array}{ccc} & \circ & \\ / & & \backslash \\ \mathscr{P} & & \mathscr{P} \end{array} + \begin{array}{ccc} & \circ & \\ / & | & \backslash \\ \mathscr{P} & \mathscr{P} & \mathscr{P} \end{array} + \cdots$$

there are several meaningful analogs of the left-sided height of binary trees:

(1) The "u-height"

$$u(\circ) = 0;$$

$$u\left(\begin{matrix}\circ\\ |\\ t\end{matrix}\right) = u(t);$$

$$u\left(\begin{matrix}\circ\\ \diagup\ \diagdown\\ t_1 \cdots t_r\end{matrix}\right) = \max\{1 + u(t_1), \ldots, 1 + u(t_{r-1}), u(t_r)\}, \qquad \text{for} \quad r \geqslant 2.$$

(2) The "v-height"

$$v(\circ) = 0;$$

$$v\left(\begin{matrix}\circ\\ \diagup\ \diagdown\\ t_1 \cdots t_r\end{matrix}\right) = \max\{r - i + v(t_i) \mid 1 \leqslant i \leqslant t\}.$$

(3) The "w-height"

$$w(\circ) = 0;$$

$$w\left(\begin{matrix}\circ\\ |\\ t\end{matrix}\right) = 1 + w(t);$$

$$w\left(\begin{matrix}\circ\\ \diagup\ \diagdown\\ t_1 \cdots t_r\end{matrix}\right) = \max\{1 + w(t_1), w(t_2), \ldots, w(t_r)\}, \qquad \text{for} \quad r \geqslant 2.$$

The asymmetric heights defined above can be interpreted as follows: u counts the maximal number of edges not being a rightmost successor of a node in a chain connecting the root with a leaf. w counts the maximal number of edges which are leftmost successors of a node in a chain connecting the root with a leaf. For example, we have for the tree t depicted in Fig. 1 the values $u(t) = 2$, $v(t) = 4$, and $w(t) = 3$.

A short reflection tells us that u determines the recursion depth of the optimized tree traversal algorithm. (The non-optimized algorithm

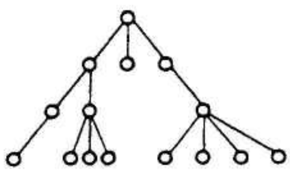
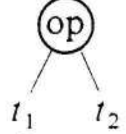


FIGURE 1

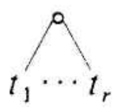
corresponds to the treatment of (de Bruijn, Knuth, and Rice, 1972).) The interpretation of v is a bit more complex:

Recall that a binary tree can be used to represent arithmetic expressions; a simple strategy of the evaluation is "from right to left", i.e., to evaluate



we evaluate t_2, use one register to keep that value, evaluate t_1, and then perform "op." It is clear that the maximal number of registers during the evaluation of a binary tree is exactly h^*.

Planted plane trees are well suited to encode arithmetic expressions where k-ary operations may occur for any k. The same strategy as in the case of binary trees leads to v (evaluating



from right to left during the consideration of t_i already $r-i$ registers are used to keep intermediate values).

The interest in w originates from another source; however, since this parameter fits well into the concept of asymmetric heights we have have decided to include it into our discussion.

In any instance we are interested in the average value u_n, v_n, and w_n of the "height" u, v, and w of the trees in $\mathscr{P}_n$, i.e., the trees of size n in $\mathscr{P}$. Our main results are:

THEOREM 1.1. (a) $u_n = \frac{1}{2}\sqrt{\pi n} - 1 + \mathcal{O}(n^{-1/2})$

(b) $v_n = \sqrt{\pi n} - \frac{5}{2} + \mathcal{O}(n^{-1/2})$

(c) $w_n = \frac{1}{2}\sqrt{\pi n} + \mathcal{O}(n^{1/4+\eta})$, *for all* $\eta > 0$.

The method to achieve these results may be described in short as follows: We start by defining appropriate generating functions refering to trees of a specified height. In cases (a) and (b) it is possible to relate the generating functions in question to similar ones occurring in the analysis of (de Bruijn, Knuth, and Rice, 1972). Since we know more or less explicit expressions for these quantities, it is possible to study the analytic behavior, in particular, the local behavior near the singularities. On the contrary, it seems to be impossible to find similar pseudoexplicit formulae in case (c), however, the analytic behavior of an appropriate generating function can also be found by a more delicate approach.

In all cases the final step is to translate this "local information" on a generating function into the asymptotic behavior of its coefficients by means of a "translation lemma" due to (Flajolet and Odlyzko, 1982).

The paper is concluded by a section applying some of the methods and results outlined above to a similar but slightly different topic concerning pairs of lattice paths (or "animals").

2. The U-Height

Let $P_h(z)$ (resp. $U_h(z)$) be the generating functions of trees in $\mathscr{P}$ with ordinary height (resp. asymmetric "height" u) $\leqslant h$ and

$$y(z)=\frac{1-\sqrt{1-4z}}{2} \tag{1}$$

as the generating function of all trees in $\mathscr{P}$.

In order to get the generating functions of "heights" of trees of equal size we observe that

$$\sum_{h\geqslant 1} h(P_h(z)-P_{h-1}(z))=\sum_{h\geqslant 0}(y(z)-P_h(z)) \tag{2}$$

as well as

$$\sum_{h\geqslant 1} h(U_h(z)-U_{h-1}(z))=\sum_{h\geqslant 0}(y(z)-U_h(z)). \tag{3}$$

Considering the defining symbolic equation for the family $\mathscr{P}$ in Section 1, the recursion

$$P_0(z)=z; \qquad P_h(z)=\frac{z}{1-P_{h-1}(z)} \tag{4}$$

is obtained; compare (de Bruijn, Knuth, and Rice, 1972). In the same paper it is shown that

$$P_h(z)=\frac{x}{1+x}\cdot\frac{1-x^{h+1}}{1-x^{h+2}}, \qquad \text{where} \quad z=\frac{x}{(1+x)^2}. \tag{5}$$

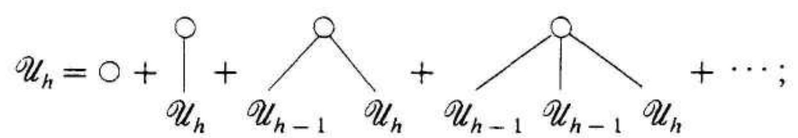
In the following proposition we exhibit a connection between the functions U_h and P_h:

Proposition 2.1. $U_h(z)=P_{2h+1}(z)$.

Proof 1 (analytic argument). We have

$$U_0(z)=\frac{z}{1-z}.$$

In order to get a recurrence relation for $U_h(z)$ we observe that the following symbolic equations hold for the families $\mathscr{U}_h$ of trees t in $\mathscr{P}$ with $u(t)\leqslant h$:



hence

$$U_h(z) = z + \frac{zU_h(z)}{1 - U_{h-1}(z)},$$

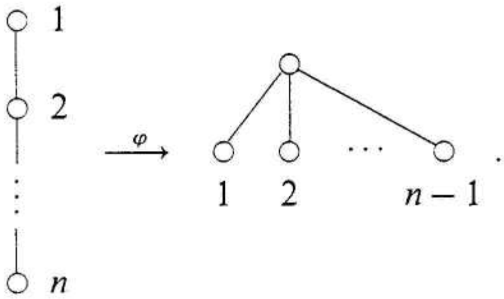
so that

$$U_h(z) = z \Bigg/ \left(1 - \frac{z}{1 - U_{h-1}(z)}\right).$$

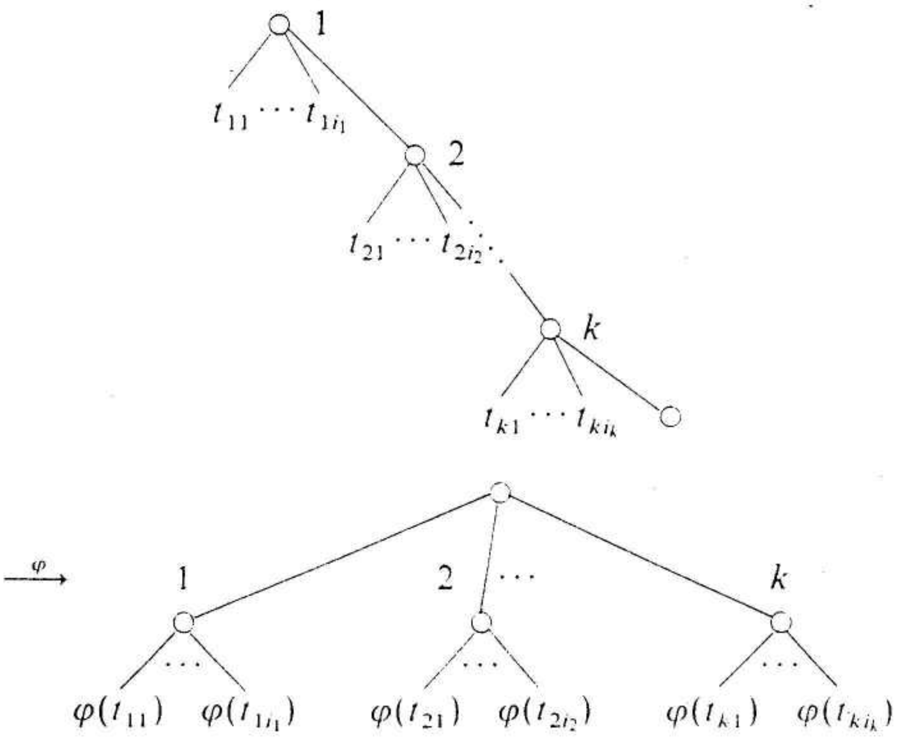
From the last expression, the lemma follows inductively by a comparison with (4).

Proof 2 (combinatorial argument). In this proof we establish a bijection φ between the family $\mathscr{U}_h$ of trees t in $\mathscr{P}$ with $u(t) \leqslant h$ and the family $\mathscr{P}_{2h+1}$ of trees t with ordinary height $\leqslant 2h+1$ in the following way:

$\varphi\colon \mathscr{U}_0 \to \mathscr{P}_1$



Then, recursively, for $t \in \mathscr{U}_h$ with subtrees $t_{rs} \in \mathscr{U}_{h-1}$:



The idea of the construction of φ is to bring (recursively) the rightmost successors to the same level as their predecessors. An inverse of φ is obtained by reversing the recursive process in the definition of φ. Now it is easy to prove by induction that φ maps $\mathscr{U}_h$ onto $\mathscr{P}_{2h+1}$. ∎

We are now ready to establish the local expansion of the generating function $\sum_{h \geqslant 0} (y - U_h)$ (compare (3)) about its singularity closest to the origin:

PROPOSITION 2.2. *With $\mu(z) = 1 - 4z$ and some constants K_1, K_2 we have in a neighborhood $\{z \in \mathbb{C} : 0 < |z - \frac{1}{4}| < \delta,\ z - \frac{1}{4} \notin \mathbb{R}^+\}$ of $\frac{1}{4}$:*

$$\sum_{h \geqslant 0} (y - U_h) = -\tfrac{1}{8} \log \mu + K_1 + \tfrac{1}{2}\mu^{1/2} + K_2 \mu + \cdots$$

Proof. By Proposition 2.1,

$$\sum_{h \geqslant 0} (y - U_h) = \sum_{h \geqslant 0} (y - P_{2h+1})$$

and by substitution from (5)

$$= \sum_{h \geqslant 0} \left(\frac{x}{1+x} - \frac{x}{1+x} \cdot \frac{1 - x^{2h+2}}{1 - x^{2h+3}} \right)$$

$$= -\frac{x}{1+x} + \frac{1-x}{1+x} \sum_{h \geqslant 0} \frac{x^{2h+1}}{1 - x^{2h+1}}.$$

The last series may be rewritten as

$$\sum_{h \geqslant 0} \frac{x^{2h+1}}{1 - x^{2h+1}} = \sum_{k \geqslant 1} d_1(k)\, x^k,$$

where $d_1(k)$ denotes the number of all odd divisors of k. Denoting by $d_2(k)$ (resp. $d(k)$) the number of all even (resp. all) divisors of k we have

$$d_1(k) = d(k) - d_2(k)$$

and

$$d_2(2k) = d(k), \qquad d_2(2k+1) = 0,$$

so that

$$\sum_{h \geqslant 0} \frac{x^{2h+1}}{1 - x^{2h+1}} = \sum_{k \geqslant 1} d(k)\, x^k - \sum_{k \geqslant 1} d(k)\, x^{2k}.$$

Thus we have

$$\sum_{h\geqslant 0}(y-U_h)=-\frac{x}{1+x}+\frac{1-x}{1+x}\sum_{k\geqslant 1}d(k)\,x^k-\frac{1-x^2}{(1+x)^2}\sum_{k\geqslant 1}d(k)\,x^{2k}.$$

It is shown in (Prodinger, 1984) that

$$\frac{1-x}{1+x}\sum_{k\geqslant 1}d(k)\,x^k=-\frac{1}{4}\log\mu+K_1'+\frac{1}{4}\mu^{1/2}+K_2'\mu+\cdots. \qquad (6)$$

Since $x^2=(z/(1-2z))^2=4\mu+\mathcal{O}(\mu^2)$ it follows that

$$\frac{1-x^2}{(1+x)^2}\sum_{k\geqslant 1}d(k)\,x^{2k}=-\frac{1}{8}\log\mu+K_1''+\frac{1}{4}\mu^{1/2}+K_2''\mu+\cdots.$$

Finally,

$$\frac{x}{1+x}=y(z)=(1-\mu^{1/2})/2.$$

Combining these local expansions the proposition is established. ∎

In order to "translate" the local behavior of the generating function in question into the asymptotic behavior of its coefficients we use the following result from (Flajolet and Odlyzko, 1982):

LEMMA 2.3. *Let $F(z)$ be analytic in the domain $z\neq 1$, $|z|<\delta$, $|\mathrm{Arg}(1-z)|<\theta$, where $\delta>1$ and $\pi/2<\theta<\pi$. Assuming that inside the intersection of a neighborhood of 1 with this domain $F(z)$ allows an expansion:*

$$F(z)=A\cdot\log(1-z)+K+\sum_{i=1}^{l}a_i(1-z)^{\nu_i}+\mathcal{O}(1-z),$$

where A, K, a_i, ν_i are constants, $0<\nu_1<\nu_2<\cdots<\nu_l<1$, the coefficients F_n of the expansion $F(z)=\sum_{n\geqslant 0}F_nz^n$ have the following asymptotic behavior for $n\to\infty$:

$$F_n=-\frac{A}{n}+\sum_{i=1}^{l}\frac{a_i}{\Gamma(-\nu_i)}\cdot\frac{1}{n^{\nu_i+1}}+\mathcal{O}\left(\frac{1}{n^2}\right).$$

Applying this lemma to the local expansion of Proposition 2.2 we find

LEMMA 2.4.

$$\sum_{h\geqslant 0}(y-U_h)=\sum_{n\geqslant 0}z^n4^n\left(\frac{1}{8n}-\frac{1}{4\sqrt{\pi}}\cdot\frac{1}{n^{3/2}}+\mathcal{O}\left(\frac{1}{n^2}\right)\right).$$

Dividing by $|\mathscr{P}_n|$, which is well known to be the Catalan number

$$|\mathscr{P}_n| = \frac{1}{n}\binom{2n-2}{n-1} = \frac{1}{4\sqrt{\pi}} 4^n n^{-3/2}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right),$$

we arrive at

THEOREM 2.5. *The average value of the asymmetric height $u(t)$ of an n-node planted plane tree t is given by*

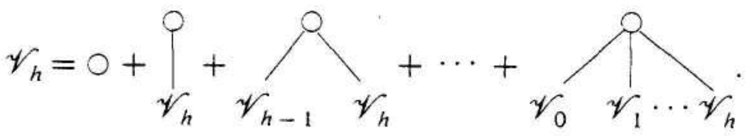
$$u_n = \tfrac{1}{2}\sqrt{\pi n} - 1 + \mathcal{O}(n^{-1/2}).$$

3. THE V-HEIGHT

In this section we perform study, analogous to that in Section 2, for the asymmetric "height" v. Let $V_h(z)$ be the generating function of trees in $\mathscr{P}$ with height $\leqslant h$. The connection between V_h and P_h is as follows:

PROPOSITION 3.1. $V_h(z) = P_{h+1}(z)$.

Proof 1 (analytic argument). We start by observing the following symbolic equation



This can be translated into the system of equations

$$V_0 = \frac{z}{1-z}$$

and

$$V_h = z + zV_h(1 + V_{h-1} + V_{h-1}V_{h-2} + \cdots + V_{h-1}\cdots V_0), \qquad h \geqslant 1.$$

From this it is an easy induction to show that

$$V_0 = \frac{z}{1-z} \qquad \text{and} \qquad V_h = \frac{z}{1 - V_{h-1}}, \qquad h \geqslant 1.$$

Since $V_0 = P_1$, a comparison with (4) finishes the proof.

Proof 2 (combinatorial argument). We establish a bijection $\psi: \mathscr{V}_h \to \mathscr{P}_{h+1}$ in the following way: In the first step we map a tree t with n nodes and

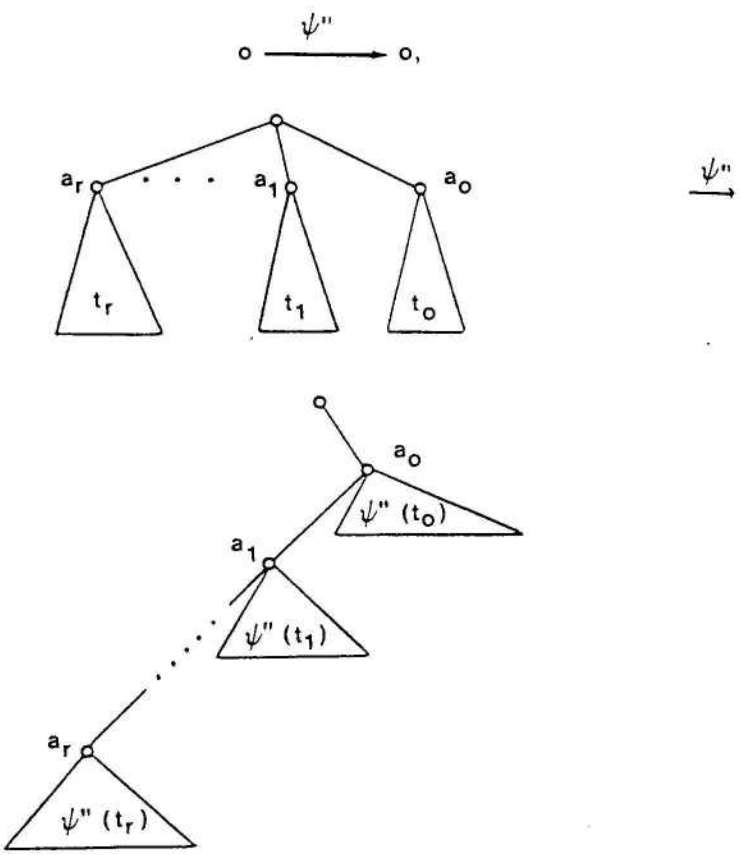


FIGURE 2

$v(t) \leqslant h$ to a binary tree $\psi'(t)$ with $n-1$ nodes and $h^*(\psi'(t)) \leqslant h$. For this purpose we use an auxiliary mapping ψ'' which is defined recursively in Fig. 2.

Note that in fact ψ' is a version of the rotation correspondence (Knuth, 1973) between binary trees and planted plane trees.

To get ψ we compose ψ' with the usual rotation correspondence between binary trees with h^*-height $\leqslant h$ and $n-1$ nodes and planted plane trees with ordinary height $\leqslant h+1$ and n nodes. ∎

THEOREM 3.2. *The average value of the asymmetric height $v(t)$ of an n-node planted plane tree t is given by*

$$v_n = \sqrt{\pi n} - \tfrac{5}{2} + \mathcal{O}(n^{-1/2}).$$

Proof. The result is an immediate consequence of Proposition 3.1 and the corresponding result (de Bruijn, Knuth, and Rice, 1972) for the ordinary height of planted plane trees:

$$h_n^* = \sqrt{\pi n} - \tfrac{3}{2} + \mathcal{O}(n^{-1/2}). \quad ∎$$

4. THE W-HEIGHT

This section is devoted to the proof of part (c) of our main theorem 1.1. While in the proofs of (a) and (b) our method was to establish an explicit connection with de Bruijn, Knith, and Rice's result for the ordinary height of planted plane trees, another approach is necessary to achieve (c). The more function-theoretic approach has been stimulated by the pioneering treatment of the average height of binary trees in (Flajolet and Odlyzko, 1982). We start with the following lemma.

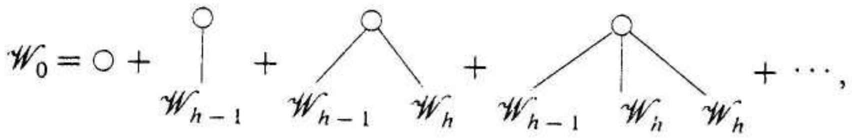
LEMMA 4.1. *Let $W_h(z)$ be the generating function of trees t in $\mathscr{P}$ with $w(t) \leqslant h$. With $\varepsilon = \sqrt{1-4z} = \sqrt{\mu}$ and $f_h(z) = y(z) - W_h(z)$:*

$$f_h^2 + (\varepsilon + z) f_h - z f_{h-1} = 0.$$

Proof. We have the system of symbolic equations

$$\mathscr{W}_0 = 0$$

and



whence

$$W_0 = z \qquad \text{and} \qquad W_h = z + \frac{z W_{h-1}}{1 - W_h}, \qquad h \geqslant 1.$$

Now we use

$$y = z + y^2 \qquad (\text{resp. } z = y(1-y)),$$

from which

$$f_h = y - W_h = y^2 = \frac{y(1-y)(y - f_{h-1})}{1 - y + f_h}$$

or

$$f_h(1 - y + f_h) = y^2(1-y) + y^2 f_h - y^2(1-y) + y(1-y) f_{h-1}.$$

Thus we get

$$f_h^2 + f_h(1 - y - y^2) - y(1-y) f_{h-1} = 0.$$

The final result is obtained by observing that

$$y = (1-\varepsilon)/2$$

and therefore

$$\varepsilon + z = 1 - 2y + y(1-y) = 1 - y - y^2. \quad \blacksquare$$

In (Flajolet and Odlyzko, 1982) a function system (e_h) fulfilling the recursion

$$e_{h+1} = (1-\varepsilon)\, e_h(1-e_h), \qquad e_0 = \tfrac{1}{2}$$

is studied in detail. At the first glance this does not seem to have any similarity with the kind of information offered on our functions f_h by Lemma 4.1. Nevertheless a well-suited rescription will allow us to proceed in a way similar to that in the paper cited above:

LEMMA 4.2. *With* $\sigma_h = f_h/(\varepsilon + z)$ *we have*:

$$\sigma_{h-1} = \frac{\varepsilon + z}{z}\, \sigma_h(1+\sigma_h), \qquad h \geqslant 1.$$

Proof. Immediate from Lemma 4.1 and the definition of σ_h. ▮

In the next step we apply a trick due to (de Bruijn, 1958, p. 157) to get the following relation.

LEMMA 4.3. *With* $\tau_h = \sigma_h^{-1}\,(z/(\varepsilon + z))^h$,

$$\tau_h = \tau_0 + \sum_{j=1}^{h} \left(\frac{z}{\varepsilon + z}\right)^j - \sum_{j=1}^{h} \left(\frac{z}{\varepsilon + z}\right)^j \frac{\sigma_j}{1+\sigma_j}.$$

Proof. Taking inverses in Lemma 4.2 we obtain

$$\begin{aligned} \sigma_{j-1}^{-1} &= \frac{z}{\varepsilon + z}\, \sigma_j^{-1}\,(1+\sigma_j)^{-1} \\ &= \frac{z}{\varepsilon + z}\, \sigma_j^{-1} - \frac{z}{\varepsilon + z} + \frac{z}{\varepsilon + z} \cdot \frac{\sigma_j}{1+\sigma_j}. \end{aligned}$$

Multiplying this equation by $(z/(\varepsilon + z))^{j-1}$ we get

$$\tau_{j-1} = \tau_j - \left(\frac{z}{\varepsilon + z}\right)^j + \left(\frac{z}{\varepsilon + z}\right)^j \frac{\sigma_j}{1+\sigma_j}.$$

Summing up yields the announced formula. ▮

The last result has a reasonable similarity with Lemma 5 of (Flajolet and Odlyzko, 1982), which is a corresponding result for the functions e_h cited above. For this reason the further analysis of the function series $\sum_h f_h$ can be carried out close to the lines of Flajolet and Odlyzko's paper, so that we confine ourselves to point out the key steps:

First we observe that the relation of Lemma 4.3 suggests

$$L(z) = \frac{\varepsilon(\varepsilon + z)}{z} \sum_{h \geq 1} \left(\frac{z}{\varepsilon + z}\right)^h \Bigg/ \left(1 - \left(\frac{z}{\varepsilon + z}\right)^h\right)$$

as a good approximation of $\sum_{h \geq 1} f_h$. In fact the difference

$$D(z) = \sum_{h \geq 0} f_h(z) - L(z) \tag{7}$$

allows the following estimation (compare the proof of Lemma 9 in (Flajolet and Odlyzko, 1982)):

LEMMA 4.4. *For z in a neighborhood of $\frac{1}{4}$, with*

$$\left|z - \frac{1}{4}\right| < \alpha, \qquad \frac{\pi}{2} - \beta < \left|\operatorname{Arg}\left(z - \frac{1}{4}\right)\right| < \frac{\pi}{2} + \beta,$$

$$D(z) = D(\tfrac{1}{4}) + \mathcal{O}(|1 - 4z|^{1/4 - \eta}) \qquad \text{for any } \eta > 0.$$

It remains to find the local behaviour of $L(z)$ near $\frac{1}{4}$.

LEMMA 4.5. *For z in a neighborhood of $\frac{1}{4}$ as in Lemma* 4.4,

$$L(z) = -\tfrac{1}{4} \log \varepsilon + K_1 + \mathcal{O}(1 - 4z).$$

Proof. We substitute

$$\frac{z}{\varepsilon + z} = e^{-u}$$

to get

$$L(z) = \varepsilon e^u \sum_{h \geq 1} \frac{e^{-hu}}{1 - e^{-hu}},$$

where u is close to 0 and $|\operatorname{Arg} u|$ is close to $\pi/4$. It is shown in (Flajolet and Odlyzko, 1982) by comparing the series with an appropriate integral that

$$\sum_{h \geq 1} u \frac{e^{-hu}}{1 - e^{-hu}} = -\log u + \delta + \mathcal{O}(|u|)$$

with some constant δ. Since $u = 4\varepsilon + \mathcal{O}(\varepsilon^2)$ and $\varepsilon e^u/u = \frac{1}{4} + \mathcal{O}(\varepsilon)$ we get the desired result. ∎

Combining the two previous lemmas we obtain

LEMMA 4.6. *For z in a neighborhood of $\frac{1}{4}$ as in Lemma 4.4 and $\mu = \varepsilon^2 = 1 - 4z$ we have*

$$\sum_{h \geq 0} (y(z) - W_h(z)) = -\tfrac{1}{8} \log \mu + K + \mathcal{O}(|1 - 4z|^{1/4 - \eta})$$

for any $\eta > 0$. ∎

Now we apply the "translation lemma" 2.3 to arrive at our desired result:

THEOREM 4.7. *The average value of the asymmetric height $w(t)$ of an n-node planted plane tree t is given by*

$$w_n = \tfrac{1}{2} \sqrt{\pi n} + \mathcal{O}(n^{1/4 + \eta}) \qquad \text{for any } \eta > 0.$$ ∎

Remark. To illustrate the power of this method we mention in passing a similar problem that can be treated along the lines of this section:

Let $h_k(t)$ denote the maximal number of nodes of outdegree k in a chain connecting the root with a leaf. Furthermore let $H_{k,h}(z)$ be the generating function of the trees t with $h_k(t) \leq h$. Then we have

$$H_{k,h} = \frac{z}{1 - H_{k,h}} - zH_{k,h}^k + zH_{k,h-1}^k.$$

With $e_{k,h}(z) = y(z) - H_{k,h}(z)$ we can expand $e_{k,h}(z)$ in terms of $e_{k,h-1}(z)$ and apply de Bruijn's trick to obtain

$$\sum_{h \geq 0} e_{k,h}(z) = -\frac{k}{2^{k+3}} \log \mu + K_k + \mathcal{O}(|1 - 4z|^{1/4 - \eta}), \qquad \eta > 0.$$

As a consequence we get that the average value $h_{k,n}^*$ of the "height" $h_k(t)$ for trees t of size n is asymptotically equivalent to

$$h_{k,n}^* \sim \frac{k}{2^{k+1}} \cdot \sqrt{\pi n}, \qquad n \to \infty. \tag{8}$$

It is interesting to note that

$$\sum_{k \geq 1} h_{k,n}^* \sim h_n^* \sim \sqrt{\pi n}. \tag{9}$$

5. An Application to Pairs of Lattice Paths

A slightly different but related topic is now discussed: Following (Polya, 1969) (resp. Fürlinger and Hofbauer, 1985), we consider pairs of lattice paths in the plane, each path starting at the origin and consisting of unit horizontal and vertical steps in the positive direction.

Let $\mathscr{L}_{n,j}$ be the set of such path-pairs (π, σ) with the following properties:

(i) both π and σ end at the point $(j, n-j)$

(ii) π begins with a unit vertical step and σ with a horizontal step

(iii) π and σ do not meet between the origin and their common endpoint.

The elements of $\mathscr{L}_n = \bigcup_{j=1}^{n} \mathscr{L}_{n,j}$ are polygons with circumference $2n$, and it is well known that

$$|\mathscr{L}_n| = \frac{1}{n}\binom{2n-2}{n-1}, \qquad n \geqslant 2; \qquad |\mathscr{L}_1| = 0.$$

Now we define the height $d(\pi, \sigma)$ of a path-pair (π, σ) to be the maximal length of a "diagonal" parallel to $y = -x$ between two lattice points on the path-pair (e.g., see Fig. 3) has $d(\pi, \sigma) = 2$. Let $D_h(z)$ denote the generating function of path-pairs (π, σ) with $d(\pi, \sigma) \leqslant h$.

Proposition 5.1. $D_h(z) = P_{2h}(z) - z$.

Proof. We use the bijection between $\mathscr{L}_n$ and "Catalan" words in $\{0, 1\}^*$ described in (Fürlinger and Hofbauer, 1985): Represent a path-pair $(\pi, \sigma) \in \mathscr{L}_n$ as a sequence of pairs of steps: let v be a vertical step and h a horizontal step. The pair (π, σ) with $\pi = a_1 \cdots a_n$, $\sigma = b_1 \cdots b_n$ where each a_i and b_i is a v or h, is represented as the sequence of step-pairs $(a_1, b_1) \cdots (a_n, b_n)$. To encode the sequence of step-pairs as a Catalan word the following translation is used:

$$(v, h) \to 00 \qquad (v, v) \to 10$$
$$(h, v) \to 11 \qquad (h, h) \to 01.$$

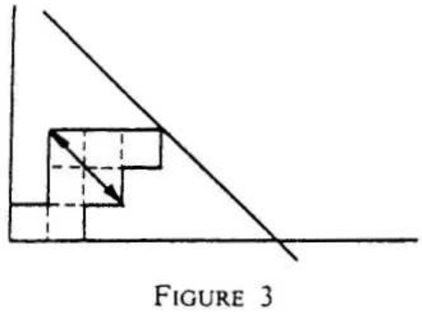


Figure 3

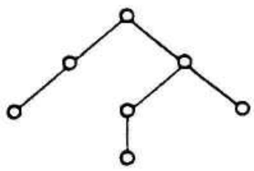


FIGURE 4

Omitting one "0" at the beginning and one "1" at the end, a Catalan word is derived. (For example: The path-pair (π, σ) from above is represented by the sequence

$$(v, h), (h, h), (v, v), (v, h), (h, v), (h, h), (h, v)$$

as encoded as the word 001100011011.)

The Catalan word is represented in the well-known way as a planted plane tree $t(\pi, \sigma)$ of size n. (See Fig. 4.)

Now we study the influence of a step-pair (a_i, b_i) of the path-pair (π, σ) on the height of the corresponding nodes of the planted plane tree $t(\pi, \sigma)$: If we had arrived at a node of height k before attaching the part of the tree corresponding to (a_i, b_i) the next two nodes will have heights

$$k-1, k \text{ if } (a_i, b_i) = (v, v) \leftrightarrow 10$$
$$k+1, k+2 \text{ if } (a_i, b_i) = (v, h) \leftrightarrow 00$$
$$k-1, k-2 \text{ if } (a_i, b_i) = (h, v) \leftrightarrow 11$$
$$k+1, k \text{ if } (a_i, b_i) = (h, h) \leftrightarrow 01.$$

On the other hand, the "local" diagonal distance l between the path-pairs develops as follows:

$$l \text{ if } (a_i, b_i) = (v, v) \leftrightarrow 0$$
$$l+1 \text{ if } (a_i, b_i) = (v, h) \leftrightarrow 00$$
$$l-1 \text{ if } (a_i, b_i) = (h, v) \leftrightarrow 11$$
$$l \text{ if } (a_i, b_i) = (h, h) \leftrightarrow 01.$$

So it is an easy consequence that the set of all path-pairs (π, σ) with $d(\pi, \sigma) = h$ corresponds to the set of trees t of size n with height t equal to

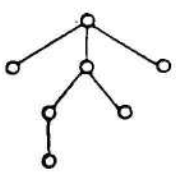


FIGURE 5

FIGURE 6

$2h-1$ or $2h$. Thus we have $D_h - D_{h-1} = P_{2h} - P_{2h-2}$, $h \geqslant 1$, with $D_0(z) = 0$. Summing up we get

$$D_h(z) = P_{2h}(z) - P_0(z) = P_{2h}(z) - z. \quad \blacksquare$$

PROPOSITION 5.2. *The average value of* $d(\pi, \sigma)$ *for path-pairs in* $\mathscr{L}_n$ *is*

$$h_n - u_n = \tfrac{1}{2}\sqrt{\pi n} - \tfrac{1}{2} + \mathcal{O}(n^{-1/2}).$$

Proof. Let $l(z) = y(z) - z$ denote the generating function of all path-pairs. Then, regarding Proposition 5.1 and 2.1,

$$\sum_{h \geqslant 0} (l - D_h) = \sum_{h \geqslant 0} (y - P_{2h}) = \sum_{h \geqslant 0} (y - P_h) - \sum_{h \geqslant 0} (y - U_h)$$

from which the result is immediate. $\blacksquare$

In (Fürlinger and Hofbauer, 1985) there is another interesting bijection between path-pairs and planted plane trees: Let $(\pi, \sigma) \in \mathscr{L}_{n,j}$ be a path-pair with steps $\pi = a_1 \cdots a_n$, $\sigma = b_1 \cdots b_n$ $(a_i, b_i \in \{v, h\})$. Now we decompose π (resp. σ) in the following way: For

$$\pi = v^{s_1}h\, v^{s_2}h \cdots v^{s_j}h, \qquad s_i \geqslant 0,$$

$$\sigma = hv^{t_1}\, hv^{t_2} \cdots hv^{t_j}, \qquad t_i \geqslant 0,$$

we consider the Catalan word

$$0^{s_1}\, 1^{t_1+1}\, 0^{s_2+1}\, 1^{t_2+1} \cdots 1^{t_{j-1}+1}\, 0^{s_j+1}\, 1^{t_j}$$

which again corresponds to a planted plane tree as usual. (In our example from above (π, σ) is encoded as 010001101101, see Fig. 5).

It is easily seen that the height of the ith leaf from the left of the tree constructed as indicated equals the area of the ith vertical rectangle of width 1

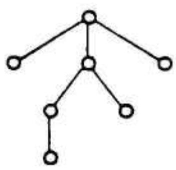


FIGURE 7

from the left between π and σ. (In our example the sequence of areas is 1, 3, 2, 1, corresponding to Fig. 6, and 1, 3, 2, 1 is also the sequence of heights of the leaves of the tree in Fig. 7.

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