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Research Article

On a Reciprocity Law for Finite Multiple Zeta Values

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It was shown by Kirschenhofer and Prodinger (1998) and Kuba et al. (2008) that harmonic numbers satisfy certain reciprocity relations, which are in particular useful for the analysis of the quickselect algorithm. The aim of this work is to show that a reciprocity relation from Kirschenhofer and Prodinger (1998) and Kuba et al. (2008) can be generalized to finite variants of multiple zeta values, involving a finite variant of the shuffle identity for multiple zeta values. We present the generalized reciprocity relation and furthermore a combinatorial proof of the shuffle identity based on partial fraction decomposition. We also present an extension of the reciprocity relation to weighted sums.

1. Introduction

Let $H_n = \sum_{k=1}^{n} 1/k$ denote the nth harmonic number and $H_n^{(s)} = \sum_{k=1}^{n} 1/k^s$ the nth harmonic number of order s, with $n, s \in \mathbb{N}$ and $H_n = H_n^{(1)}$. Kirschenhofer and Prodinger [1] analyzed the variance of the number of comparisons of the famous QUICKSELECT algorithm, also known as FIND [2] and derived a reciprocity relation for (first-order) harmonic numbers. Subsequently, the reciprocity relation of [1] was generalized [3], where the following identity was derived:

$$\sum_{k=1}^{j} \frac{H_{N-k}^{(a)}}{k^b} + \sum_{k=1}^{N+1-j} \frac{H_{N-k}^{(b)}}{k^a} = -\frac{1}{j^b (N+1-j)^a} + H_j^{(b)} H_{N+1-j}^{(a)} + R_N^{(a,b)}, \tag{1.1}$$

where $R_N^{(a,b)} = \sum_{k=1}^{N} H_{N-k}^{(a)}/k^b$, which can be evaluated into a finite analogue of the so-called Euler identity for $\zeta(a)\zeta(b)$ stated below,

$$R_N^{(a,b)} = \sum_{i=1}^{a} \binom{i+b-2}{b-1} \zeta_N(i+b-1, a+1-i) + \sum_{i=1}^{b} \binom{i+a-2}{a-1} \zeta_N(i+a-1, b+1-i), \quad (1.2)$$

where the multiple zeta values [4–9], and its finite counterpart are defined as follows:

$$\begin{aligned} \zeta(\mathbf{a}) = \zeta(a_1, \ldots, a_r) &:= \sum_{n_1 > n_2 > \cdots > n_r \geq 1} \frac{1}{n_1^{a_1} n_2^{a_2} \cdots n_r^{a_r}}, \\ \zeta_N(\mathbf{a}) = \zeta_N(a_1, \ldots, a_r) &:= \sum_{N \geq n_1 > n_2 > \cdots > n_r \geq 1} \frac{1}{n_1^{a_1} n_2^{a_2} \cdots n_r^{a_r}}. \end{aligned} \quad (1.3)$$

Note that $\zeta_N(a) = H_N^{(a)}$. Finite multiple zeta values are also called truncated multiple zeta values. They are also of great importance in particle physics, see for example the works [10–12], and closely related to so-called harmonics sums. Let $w = \sum_{i=1}^{r} a_i$ denote the weight and $d = r$ the depth of (finite) multiple zeta values. The aim of this note is to derive a generalization of the reciprocity relation (1.1), stated below in Theorem 2.1, by considering the more general sums

$$\sum_{k=1}^{j} \frac{\zeta_{k-1}(b_2, \ldots, b_s)\zeta_{N-k}(a_1, \ldots, a_r)}{k^{b_1}} + \sum_{k=1}^{N+1-j} \frac{\zeta_{k-1}(a_2, \ldots, a_r)\zeta_{N-k}(b_1, \ldots, b_s)}{k^{a_1}}, \quad (1.4)$$

instead of the previously considered sums $\sum_{k=1}^{j} H_{N-k}^{(a)}/k^b$ and $\sum_{k=1}^{N+1-j} H_{N-k}^{(b)}/k^a$. The generalization involves a finite variant of the shuffle identity for multiple zeta values; see, for example, Hoffman [13] for a general algebraic framework for shuffle products. We will give an elementary proof of the shuffle identity using only partial fraction decomposition and the combinatorial properties of the shuffle product in Sections 3.1 and 3.2. Moreover, we discuss the close relation between this finite variant of the shuffle identity and the shuffle identity for generalized polylogarithm functions; it will turn out that the finite variant of the shuffle identity is equivalent to the shuffle identity for generalized polylogarithm functions.

To simplify the presentation of this work, we will frequently use the shorthand notations $\mathbf{a} = (a_1, \ldots, a_r)$, $\mathbf{a}_2 = (a_2, \ldots, a_r)$, and $\mathbf{b} = (b_1, \ldots, b_s)$, $\mathbf{b}_2 = (b_2, \ldots, b_s)$, respectively, with $r, s \in \mathbb{N}$ and $a_i, b_k \in \mathbb{N}$ for $1 \leq i \leq r$ and $1 \leq k \leq s$.

2. Results

We will state the main theorem and two corollaries below, and subsequently discuss their proofs and the precise definition of the shuffle relation for multiple zeta values.

Theorem 2.1. *The finite multiple zeta values* $\zeta_N(\mathbf{a}) = \zeta_N(a_1,\dots,a_r)$, $\zeta_N(\mathbf{b}) = \zeta_N(b_1,\dots,b_s)$ *satisfy the following reciprocity relation.*

$$\begin{aligned}\sum_{k=1}^{j}\frac{\zeta_{k-1}(b_2,\dots,b_s)\zeta_{N-k}(a_1,\dots,a_r)}{k^{b_1}} + \sum_{k=1}^{N+1-j}\frac{\zeta_{k-1}(a_2,\dots,a_r)\zeta_{N-k}(b_1,\dots,b_s)}{k^{a_1}} \\ = \zeta_{N+1-j}(\mathbf{a})\zeta_j(\mathbf{b}) - \frac{\zeta_{j-1}(\mathbf{b}_2)\zeta_{N-j}(\mathbf{a}_2)}{j^{b_1}(N+1-j)^{a_1}} + R_N(\mathbf{a};\mathbf{b}).\end{aligned} \tag{2.1}$$

The quantity $R_N(\mathbf{a};\mathbf{b}) = \sum_{k=1}^{N}\zeta_{N-k}(\mathbf{b})\zeta_{k-1}(a_2,\dots,a_r)/k^{a_1} = R_N(\mathbf{b};\mathbf{a})$ *can be written as a sum of finite multiple zeta values, all of them having weight* $w = \sum_{i=1}^{r} a_r + \sum_{i=1}^{s} b_i$ *and depth* $d = r + s$.

Remark 2.2. The quantity $R_N(\mathbf{a};\mathbf{b})$ satisfies a shuffle identity resembling the ordinary shuffle identity for multiple zeta values $\zeta(\mathbf{a})\zeta(\mathbf{b}) = \zeta(\mathbf{a}\sqcup\!\sqcup\mathbf{b})$; see Sections 3.1, 3.2 and Proposition 3.4 for details.

Corollary 2.3. *We obtain the complementary identity*

$$\begin{aligned}&\sum_{k=1}^{j-1}\frac{\zeta_k(\mathbf{b})\zeta_{N-k-1}(\mathbf{a}_2)}{(N-k)^{a_1}} + \sum_{k=1}^{N-j}\frac{\zeta_k(\mathbf{a})\zeta_{N-k-1}(\mathbf{b}_2)}{(N-k)^{b_1}} \\ &= \frac{\zeta_{j-1}(\mathbf{b})\zeta_{N-j}(\mathbf{a}_2)}{(N+1-j)^{a_1}} + \frac{\zeta_{N-j}(\mathbf{a})\zeta_{j-1}(\mathbf{b}_2)}{j^{b_1}} - \zeta_{N+1-j}(\mathbf{a})\zeta_j(\mathbf{b}) + \frac{\zeta_{j-1}(\mathbf{b}_2)\zeta_{n-j}(\mathbf{a}_2)}{j^{b_1}(N+1-j)^{a_1}} + R_N(\mathbf{a};\mathbf{b}).\end{aligned} \tag{2.2}$$

Next we state an immediate asymptotic implication of the previous result.

Corollary 2.4. *For* $N = 2n+1$, $j = n+1$, *with* $a_1, b_1 \in \mathbb{N}\setminus\{1\}$ *and* $n \to \infty$, *we obtain the following result:*

$$\lim_{n\to\infty}\left(\sum_{k=1}^{j}\frac{\zeta_{k-1}(\mathbf{b}_2)\zeta_{N-k}(\mathbf{a})}{k^{b_1}} + \sum_{k=1}^{N+1-j}\frac{\zeta_{k-1}(\mathbf{a}_2)\zeta_{N-k}(\mathbf{b})}{k^{a_1}}\right) = 2\zeta(\mathbf{a})\zeta(\mathbf{b}). \tag{2.3}$$

3. The Proof of the Reciprocity Relation

In order to prove Theorem 2.1, we proceed as follows (using the beforehand introduced shorthand notations).

$$\begin{aligned}\sum_{k=1}^{j}\frac{\zeta_{k-1}(\mathbf{b}_2)\zeta_{N-k}(\mathbf{a})}{k^{b_1}} &= \sum_{k=1}^{j}\frac{\zeta_{k-1}(\mathbf{b}_2)}{k^{b_1}}\left(\zeta_{N-j}(\mathbf{a}) + \sum_{\ell=N+1-j}^{N-k}\frac{\zeta_{\ell-1}(\mathbf{a}_2)}{\ell^{a_1}}\right) \\ &= \zeta_{N-j}(\mathbf{a})\zeta_j(\mathbf{b}) + \sum_{k=1}^{j}\frac{\zeta_{k-1}(\mathbf{b}_2)}{k^{b_1}}\sum_{\ell=N+1-j}^{N-k}\frac{\zeta_{\ell-1}(\mathbf{a}_2)}{\ell^{a_1}}.\end{aligned} \tag{3.1}$$

After changing summations, we obtain

$$\begin{aligned}\sum_{k=1}^{j}\frac{\zeta_{k-1}(\mathbf{b}_2)\zeta_{N-k}(\mathbf{a})}{k^{b_1}} &= \zeta_{N-j}(\mathbf{a})\zeta_j(\mathbf{b}) + \sum_{\ell=N+1-j}^{N-1}\frac{\zeta_{\ell-1}(\mathbf{a}_2)}{\ell^{a_1}}\sum_{k=1}^{N-\ell}\frac{\zeta_{k-1}(\mathbf{b}_2)}{k^{b_1}} \\ &= \zeta_{N-j}(\mathbf{a})\zeta_j(\mathbf{b}) + \sum_{\ell=N+1-j}^{N-1}\frac{\zeta_{\ell-1}(\mathbf{a}_2)\zeta_{N-\ell}(\mathbf{b})}{\ell^{a_1}}.\end{aligned} \tag{3.2}$$

Using

$$\begin{aligned}\zeta_{N-j}(\mathbf{a})\zeta_j(\mathbf{b}) + \frac{\zeta_{N-j}(\mathbf{a}_2)\zeta_{j-1}(\mathbf{b})}{(N+1-j)^{a_1}} &= \zeta_{N-j}(\mathbf{a})\zeta_j(\mathbf{b}) + \frac{\zeta_{N-j}(\mathbf{a}_2)}{(N+1-j)^{a_1}}\left(\zeta_j(\mathbf{b}) - \frac{\zeta_{j-1}(\mathbf{b}_2)}{j^{b_1}}\right) \\ &= \zeta_{N+1-j}(\mathbf{a})\zeta_j(\mathbf{b}) - \frac{\zeta_{N-j}(\mathbf{a}_2)\zeta_{j-1}(\mathbf{b}_2)}{(N+1-j)^{a_1}j^{b_1}},\end{aligned} \tag{3.3}$$

and the fact that $\zeta_0(\mathbf{b}) = 0$ gives the intermediate result

$$\sum_{k=1}^{j}\frac{\zeta_{k-1}(\mathbf{b}_2)\zeta_{N-k}(\mathbf{a})}{k^{b_1}} = \zeta_{N+1-j}(\mathbf{a})\zeta_j(\mathbf{b}) - \frac{\zeta_{N-j}(\mathbf{a}_2)\zeta_{j-1}(\mathbf{b}_2)}{(N+1-j)^{a_1}j^{b_1}} + \sum_{\ell=N+2-j}^{N}\frac{\zeta_{\ell-1}(\mathbf{a}_2)\zeta_{N-\ell}(\mathbf{b})}{\ell^{a_1}}, \tag{3.4}$$

Add the sum $\sum_{k=1}^{N+1-j}\zeta_{k-1}(\mathbf{a}_2)\zeta_{N-k}(\mathbf{b})/k^{a_1}$ to both sides of the equation above. This proves the first part of Theorem 2.1 and

$$R_N(\mathbf{a};\mathbf{b}) = \sum_{k=1}^{N}\frac{\zeta_{N-k}(\mathbf{b})\zeta_{k-1}(a_2,\dots,a_r)}{k^{a_1}}. \tag{3.5}$$

For the evaluation of $R_N(\mathbf{a};\mathbf{b})$, we note that $R_0(\mathbf{a};\mathbf{b}) = 0$, and further

$$R_N(\mathbf{a};\mathbf{b}) = \sum_{k=1}^{N}(R_k(\mathbf{a};\mathbf{b}) - R_{k-1}(\mathbf{a};\mathbf{b})). \tag{3.6}$$

Since $\zeta_0(\mathbf{b}) = 0$, we have

$$\begin{aligned}R_N(\mathbf{a};\mathbf{b}) - R_{N-1}(\mathbf{a};\mathbf{b}) &= \sum_{k=1}^{N}\frac{\zeta_{N-k}(\mathbf{b})\zeta_{k-1}(a_2,\dots,a_r)}{k^{a_1}} - \sum_{k=1}^{N-1}\frac{\zeta_{N-1-k}(\mathbf{b})\zeta_{k-1}(a_2,\dots,a_r)}{k^{a_1}} \\ &= \sum_{k=1}^{N-1}\frac{(\zeta_{N-k}(\mathbf{b}) - \zeta_{N-1-k}(\mathbf{b})\zeta_{k-1}(a_2,\dots,a_r)}{k^{a_1}} \\ &= \sum_{k=1}^{N-1}\frac{\zeta_{N-1-k}(b_2,\dots,b_s)\zeta_{k-1}(a_2,\dots,a_r)}{(N-k)^{b_1}k^{a_1}}.\end{aligned} \tag{3.7}$$

Now we use the following partial fraction decomposition (This identity has been rediscovered many times. For a fascinating historic account, see [14].) , which appears already in [15],

$$\frac{1}{k^a(N-k)^b} = \sum_{i=1}^{a} \frac{\binom{i+b-2}{b-1}}{N^{i+b-1}k^{a+1-i}} + \sum_{i=1}^{b} \frac{\binom{i+a-2}{a-1}}{N^{i+a-1}(N-k)^{b+1-i}}, \tag{3.8}$$

and obtain

$$\begin{aligned}\sum_{k=1}^{N-1} \frac{\zeta_{k-1}(a_2,\ldots,a_r)\zeta_{N-1-k}(b_2,\ldots,b_s)}{(N-k)^{b_1}k^{a_1}} &= \sum_{i=1}^{a_1}\sum_{k=1}^{N-1} \frac{\binom{i+b_1-2}{b_1-1}\zeta_{k-1}(a_2,\ldots,a_r)\zeta_{N-1-k}(b_2,\ldots,b_s)}{N^{i+b_1-1}k^{a_1+1-i}} \\ &+ \sum_{i=1}^{b_1}\sum_{k=1}^{N-1} \frac{\binom{i+a_1-2}{a_1-1}\zeta_{k-1}(a_2,\ldots,a_r)\zeta_{N-1-k}(b_2,\ldots,b_s)}{N^{i+a_1-1}(N-k)^{b_1+1-i}}.\end{aligned} \tag{3.9}$$

Consequently, by summing up according to (3.6), we get the following recurrence relation for $R_N(\mathbf{a};\mathbf{b})$:

$$\begin{aligned}R_N(\mathbf{a};\mathbf{b}) &= \sum_{i=1}^{a_1}\sum_{n_1=1}^{N} \frac{\binom{i+b_1-2}{b_1-1}}{n_1^{i+b_1-1}} R_{n_1-1}(a_1+1-i, a_2,\ldots,a_r; b_2,\ldots,b_s) \\ &+ \sum_{i=1}^{b_1}\sum_{n_1=1}^{N} \frac{\binom{i+a_1-2}{a_1-1}}{n_1^{i+a_1-1}} R_{n_1-1}(a_2,\ldots,a_r; b_1+1-i, b_2,\ldots,b_s).\end{aligned} \tag{3.10}$$

This recurrence relation suggests that there exists an evaluation of $R_N(\mathbf{a};\mathbf{b})$ into sums of finite multiple zeta values, all of them having weight $w = \sum_{i=1}^{r} a_r + \sum_{i=1}^{s} b_i$ and depth $d = r + s$. In order to specify this evaluation, we need to introduce the shuffle product for words over a noncommutative alphabet and to study the arising shuffle algebra, and its relation to (finite) multiple zeta values and $R_N(\mathbf{a};\mathbf{b})$. For a general algebraic framework for the shuffle product, we refer the reader to the work of Hoffman [13]. We remark that the recurrence relation above for $R_N(\mathbf{a};\mathbf{b})$ was already derived in the context of particle physics [11, 12]. Furthermore, weighted extensions including alternating sign versions have been treated there. An important algorithmic treatment of such sums is implemented in the package `Summer` for the computer algebra system `Form`.

3.1. The Shuffle Algebra

Let $\mathcal{A}$ denote a finite noncommutative alphabet consisting of a set of letters. A word $\mathbf{w}$ on the alphabet $\mathcal{A}$ consists of a sequence of letters from $\mathcal{A}$. Let $\mathcal{A}^*$ denote the set of all words on the alphabet $\mathcal{A}$. A polynomial on $\mathcal{A}$ over $\mathbb{Q}$ is a rational linear combination of words on

$\mathcal{A}$. The set of all such polynomials is denoted by $\mathbb{Q}\langle\mathcal{A}\rangle$. Let the shuffle product of two words $\mathbf{w}, \mathbf{v} \in \mathcal{A}^*$, with $\mathbf{w} = x_1 \cdots x_n$, $\mathbf{v} = x_{n+1} \cdots x_{n+m}$, $x_i \in \mathcal{A}$ for $1 \le i \le n+m$, be defined as follows:

$$\mathbf{w} ⧢ \mathbf{v} := \sum x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n+m)}, \tag{3.11}$$

where the sum runs over all $\binom{n+m}{n}$ permutations $\sigma \in \mathfrak{S}_{n+m}$ which satisfy $\sigma^{-1}(j) < \sigma^{-1}(k)$ for all $1 \le j < k \le n$ and $n+1 \le j < k \le n+m$. Note that the sum runs over all words of length $n+m$, counting multiplicities, in which the relative orders of the letters $x_1, \dots, x_n$ and $x_{n+1}, \dots, x_{n+m}$ are preserved. Equivalently, the shuffle product of two words $\mathbf{w}, \mathbf{v} \in \mathcal{A}^*$ can be defined in a recursive way:

$$\begin{gathered} \forall \mathbf{w} \in \mathcal{A}^*, \quad \varepsilon ⧢ \mathbf{w} = \mathbf{w} ⧢ \varepsilon = \mathbf{w}, \\ \forall x, y \in \mathcal{A},\ \mathbf{w}, \mathbf{v} \in \mathcal{A}^*, \quad x\mathbf{w} ⧢ y\mathbf{v} = x(\mathbf{w} ⧢ y\mathbf{v}) + y(x\mathbf{w} ⧢ \mathbf{v}). \end{gathered} \tag{3.12}$$

The shuffle product extends to $\mathbb{Q}\langle\mathcal{A}\rangle$ by linearity. Note that the set $\mathbb{Q}\langle\mathcal{A}\rangle$, provided with the shuffle product ⧢, becomes a commutative and associative algebra. We remark that the term "shuffle" is used because such permutations arise in riffle shuffling a deck of $n+m$ cards cut into one pile of n cards and a second pile of m cards [7].

In the following, we will restrict ourselves to the non-commutative alphabet $\mathcal{A} = \{\omega_0, \omega_1\}$ and the arising shuffle algebra $(\mathbb{Q}\langle\mathcal{A}\rangle, ⧢)$. Hoang and Petitot [16] derived a shuffle identity for words $A = \omega_0^{a-1}\omega_1$, $B = \omega_0^{b-1}\omega_1$, which is stated below.

Lemma 3.1. *For $a, b \in \mathbb{N}$, let $A = \omega_0^{a-1}\omega_1$ and $B = \omega_0^{b-1}\omega_1$ be words on the non-commutative alphabet $\mathcal{A} = \{\omega_0, \omega_1\}$.*

$$A ⧢ B = \sum_{i=0}^{a-1} \binom{b-1+i}{b-1} \omega_0^{b-1+i}\omega_1\omega_0^{a-1-i}\omega_1 + \sum_{i=0}^{b_1-1} \binom{a-1+i}{a-1} \omega_0^{a-1+i}\omega_1\omega_0^{b-1-i}\omega_1. \tag{3.13}$$

We will use a slight extension of this identity, which easily follows from the recursive definition of the shuffle product.

Lemma 3.2. *For $r, s \ge 1$ and $a_i, b_j \in \mathbb{N}$, $1 \le i \le r$, $1 \le j \le s$, let $A := \omega_0^{a_1-1}\omega_1 \cdots \omega_0^{a_r-1}\omega_1$ and $B := \omega_0^{b_1-1}\omega_1 \cdots \omega_0^{b_s-1}\omega_1$ be words on the non-commutative alphabet $\mathcal{A} = \{\omega_0, \omega_1\}$.*

$$A ⧢ B = \sum_{i=1}^{a_1} \binom{i+b_1-2}{b_1-1} \omega_0^{i+b_1-2}\omega_1 \left(A_i' ⧢ B_2\right) + \sum_{i=1}^{b_1} \binom{i+a_1-2}{a_1-1} \omega_0^{i+a_1-2}\omega_1 \left(A_2 ⧢ B_i'\right), \tag{3.14}$$

with $A_i' := \omega_0^{a_1-i}\omega_1\omega_0^{a_2-1}\omega_1 \cdots \omega_0^{a_r-1}\omega_1$, $B_i' := \omega_0^{b_1-i}\omega_1\omega_0^{b_2-1}\omega_1 \cdots \omega_0^{b_s-1}\omega_1$ and further $A_2 := \omega_0^{a_2}\omega_1 \cdots \omega_0^{a_r-1}\omega_1$, $B_2 := \omega_0^{b_2-1}\omega_1 \cdots \omega_0^{b_s-1}\omega_1$.

Note that the partial fraction decomposition (3.8) of $1/k^a(N-k)^b$ somewhat mimics the shuffle identity for words $A = \omega_0^{a-1}\omega_1$, $B = \omega_0^{b-1}\omega_1$, derived by Hoang and Petitot [16].

3.2. The Shuffle Algebra and Finite Multiple Zeta Values

Let $\mathbf{a}$ denote an arbitrary r-tuple of positive integers $\mathbf{a} = (a_1, \ldots, a_r)$ with $a_i \in \mathbb{N}$ for $1 \le i \le r$ and $r \ge 1$. To any $\mathbf{a}$, we will associate a unique word $A = A(\mathbf{a})$ over the non-commutative alphabet $\mathcal{A} = \{\omega_0, \omega_1\}$ as follows: $A = A(\mathbf{a})$ such that $A := \omega_0^{a_1-1}\omega_1\omega_0^{a_2-1}\omega_1 \cdots \omega_0^{a_r-1}\omega_1$. Let $\mathcal{A}^*$ denote the set of all words over the alphabet $\mathcal{A}$. Let $(Z_N)_{N\ge 1}$ denote a family of linear maps from the algebra $\mathbb{Q}\langle\mathcal{A}\rangle$ to the rational numbers, $Z_N : \mathbb{Q}\langle\mathcal{A}\rangle \to \mathbb{Q}$, mapping words over the non-commutative alphabet $\mathcal{A} = \{\omega_0, \omega_1\}$ to finite multiple zeta values in the following way. For words $A := \omega_0^{a_1-1}\omega_1\omega_0^{a_2-1}\omega_1 \cdots \omega_0^{a_r-1}\omega_1 \in \mathcal{A}^*$, with $r, N \ge 1$, we define

$$Z_N(A) = Z_N\left(\omega_0^{a_1-1}\omega_1\omega_0^{a_2-1}\omega_1 \cdots \omega_0^{a_r-1}\omega_1\right) = \zeta_N(a_1, \ldots, a_r) = \zeta_N(\mathbf{a}). \tag{3.15}$$

Moreover, we additionally define $Z_0(A) = \zeta_0(\mathbf{a}) = 0$ for all $A \in \mathcal{A}^*$, and $Z_N(\varepsilon) = 1$ for all $N \ge 1$. The family of maps $(Z_N)_{N\ge 1}$ linearly extend to $\mathbb{Q}\langle\mathcal{A}\rangle$. By the recursive definition of the finite multiple zeta values, we can express the images of the maps Z_N in a recursive way. Let $A := \omega_0^{a_1-1}\omega_1\omega_0^{a_2-1}\omega_1 \cdots \omega_0^{a_r-1}\omega_1 \in \mathcal{A}^*$, with $r \ge 1$ and $a_1, \ldots, a_r \ge 1$.

$$Z_N(A) = \zeta_N(\mathbf{a}) = \sum_{n_1=1}^{N} \frac{1}{n_1^{a_1}} \zeta_{n_1-1}(a_2, \ldots, a_r) = \sum_{n_1=1}^{N} \frac{1}{n_1^{a_1}} Z_{n_1-1}\left(\omega_0^{a_2-1}\omega_1 \cdots \omega_0^{a_r-1}\omega_1\right). \tag{3.16}$$

We need the following result.

Lemma 3.3. *For $r, s \ge 1$ and $a_i, b_j \in \mathbb{N}$, $1 \le i \le r$, $1 \le j \le s$, let $A := \omega_0^{a_1-1}\omega_1 \cdots \omega_0^{a_r-1}\omega_1$ and $B := \omega_0^{b_1-1}\omega_1 \cdots \omega_0^{b_s-1}\omega_1$ be words on the non-commutative alphabet $\mathcal{A} = \{\omega_0, \omega_1\}$. Then,*

$$Z_N(A \sqcup\!\sqcup B) = \sum_{i=1}^{a_1}\sum_{n_1=1}^{N} \frac{\binom{i+b_1-2}{b_1-1}}{n_1^{i+b_1-1}} Z_{n_1-1}\left(A_i' \sqcup\!\sqcup B_2\right) + \sum_{i=1}^{b_1}\sum_{n_1=1}^{N} \frac{\binom{i+a_1-2}{a_1-1}}{n_1^{i+a_1-1}} Z_{n_1-1}\left(A_2 \sqcup\!\sqcup B_i'\right). \tag{3.17}$$

The depths $d = r + s$ and the weights $w = \sum_{i=1}^{r} a_i + \sum_{k=1}^{s} b_k$ of the arising finite multiple zeta values are all the same.

Proof. By linearity of the maps Z_N and Lemma 3.2, we get first

$$\begin{aligned} Z_N(A \sqcup\!\sqcup B) = &\sum_{i=1}^{a_1} \binom{i+b_1-2}{b_1-1} Z_N\left(\omega_0^{i+b_1-2}\omega_1\left(A_i' \sqcup\!\sqcup B_2\right)\right) \\ &+ \sum_{i=1}^{b_1} \binom{i+a_1-2}{a_1-1} Z_N\left(\omega_0^{i+a_1-2}\omega_1\left(A_2 \sqcup\!\sqcup B_i'\right)\right), \end{aligned} \tag{3.18}$$

using the notations of Lemma 3.2 for A_i', B_i', A_2, B_2. By definition of the shuffle product, $A_i' \sqcup\!\sqcup B_2 \in \mathbb{Q}\langle\mathcal{A}\rangle$ and $A_2 \sqcup\!\sqcup B_i' \in \mathbb{Q}\langle\mathcal{A}\rangle$ are rational linear combinations of words over $\mathcal{A}$. Let

$\{A_i' ⧢ B_2\}$ and $\{A_2 ⧢ B_i'\}$ denote the *sets* of different words generated by the shuffles $A_i' ⧢ B_2$ and $A_2 ⧢ B_i'$. Using the set notation, we write

$$A_i' ⧢ B_2 = \sum_{\mathbf{w}\in\{A_i' ⧢ B_2\}} q_{\mathbf{w}}\mathbf{w}, \qquad A_2 ⧢ B_i' = \sum_{\mathbf{w}\in\{A_2 ⧢ B_i'\}} q_{\mathbf{w}}\mathbf{w}, \tag{3.19}$$

with $q_{\mathbf{w}} \in \mathbb{Q}$ and $\mathbf{w} \in \mathcal{A}^*$, which helps to obtain a simple presentation of the subsequent calculations. We have

$$\begin{aligned} Z_N(A ⧢ B) = {} & \sum_{i=1}^{a_1}\binom{i+b_1-2}{b_1-1} Z_N\left(\omega_0^{i+b_1-2}\omega_1 \sum_{\mathbf{w}\in\{A_i' ⧢ B_2\}} q_{\mathbf{w}}\mathbf{w}\right) \\ & + \sum_{i=1}^{b_1}\binom{i+a_1-2}{a_1-1} Z_N\left(\omega_0^{i+a_1-2}\omega_1 \sum_{\mathbf{w}\in\{A_2 ⧢ B_i'\}} q_{\mathbf{w}}\mathbf{w}\right). \end{aligned} \tag{3.20}$$

Using the linearity of the maps Z_N and the fact that we can recursively describe their images, we get further

$$\begin{aligned} Z_N(A ⧢ B) = {} & \sum_{i=1}^{a_1}\binom{i+b_1-2}{b_1-1} \sum_{\mathbf{w}\in\{A_i' ⧢ B_2\}} q_{\mathbf{w}} \sum_{n_1=1}^{N} \frac{1}{n_1^{i+b_1-1}} Z_{n_1-1}(\mathbf{w}) \\ & + \sum_{i=1}^{b_1}\binom{i+a_1-2}{a_1-1} \sum_{\mathbf{w}\in\{A_2 ⧢ B_i'\}} q_{\mathbf{w}} \sum_{n_1=1}^{N} \frac{1}{n_1^{i+a_1-1}} Z_{n_1-1}(\mathbf{w}). \end{aligned} \tag{3.21}$$

Interchanging the latter summations gives the stated result.

$$\begin{aligned} Z_N(A ⧢ B) = {} & \sum_{i=1}^{a_1}\binom{i+b_1-2}{b_1-1} \sum_{n_1=1}^{N} \frac{1}{n_1^{i+b_1-1}} \sum_{\mathbf{w}\in\{A_i' ⧢ B_2\}} q_{\mathbf{w}} Z_{n_1-1}(\mathbf{w}) \\ & + \sum_{i=1}^{b_1}\binom{i+a_1-2}{a_1-1} \sum_{n_1=1}^{N} \frac{1}{n_1^{i+a_1-1}} \sum_{\mathbf{w}\in\{A_2 ⧢ B_i'\}} q_{\mathbf{w}} Z_{n_1-1}(\mathbf{w}) \\ = {} & \sum_{i=1}^{a_1}\sum_{n_1=1}^{N} \frac{\binom{i+b_1-2}{b_1-1}}{n_1^{i+b_1-1}} Z_{n_1-1}(A_i' ⧢ B_2) + \sum_{i=1}^{b_1}\sum_{n_1=1}^{N} \frac{\binom{i+a_1-2}{a_1-1}}{n_1^{i+a_1-1}} Z_{n_1-1}(A_2 ⧢ B_i'). \end{aligned} \tag{3.22}$$

It can easily be checked that the finite multiple zeta values all have the same depth and weight. $\square$

Now we are ready to provide the evaluation of $R_N(\mathbf{a};\mathbf{b})$.

Proposition 3.4. *For arbitrary $r,s\geq 1$, let $\mathbf{a}$ and $\mathbf{b}$ be given by $\mathbf{a}=(a_1,\dots,a_r)$ and $\mathbf{b}=(b_1,\dots,b_s)$, with $a_i,b_j\in\mathbb{N}$ for $1\leq i\leq r$, $1\leq j\leq s$. Let $A=A(\mathbf{a})$ and $B=A(\mathbf{b})$ denote the words associated to $\mathbf{a}$ and $\mathbf{b}$ by $A:=\omega_0^{a_1-1}\omega_1\cdots\omega_0^{a_r-1}\omega_1$ and $B:=\omega_0^{b_1-1}\omega_1\cdots\omega_0^{b_s-1}\omega_1$. Then, for arbitrary $N\geq 1$,*

$$R_N(\mathbf{a};\mathbf{b})=Z_N(A\sqcup\!\sqcup B). \tag{3.23}$$

Proof. We use induction with respect to $d=r+s$, corresponding to the depths of the arising finite multiple zeta values. The result clearly holds for depth $d=2$; see identity (1.2), as shown in [3]. Now assume that $d\geq 3$. Using the recurrence relation (3.10) for $R_N(\mathbf{a};\mathbf{b})$, we get

$$\begin{aligned} R_N(\mathbf{a};\mathbf{b}) &= \sum_{i=1}^{a_1}\sum_{n_1=1}^{N}\frac{\binom{i+b_1-2}{b_1-1}}{n_1^{i+b_1-1}}R_{n_1-1}(a_1+1-i,a_2,\dots,a_r;b_2,\dots,b_s)\\ &\quad+\sum_{i=1}^{b_1}\sum_{n_1=1}^{N}\frac{\binom{i+a_1-2}{a_1-1}}{n_1^{i+a_1-1}}R_{n_1-1}(a_2,\dots,a_r;b_1+1-i,b_2,\dots,b_s). \end{aligned} \tag{3.24}$$

The induction hypothesis states that $R_N(\mathbf{a};\mathbf{b})=Z_N(A\sqcup\!\sqcup B)$ for arbitrary $r,s\geq 1$ such that $r+s<d$ and arbitrary $N\geq 1$. By the recurrence relation for $R_N(\mathbf{a};\mathbf{b})$, we can reduce $R_N(\mathbf{a};\mathbf{b})$ to values of the types $R_{n_1-1}(a_1+1-i,a_2,\dots,a_r;b_2,\dots,b_s)$ and $R_{n_1-1}(a_2,\dots,a_r;b_1+1-i,b_2,\dots,b_s)$, which are of depth smaller than $d=r+s$. Hence, we get by the induction hypothesis

$$R_N(\mathbf{a};\mathbf{b})=\sum_{i=1}^{a_1}\sum_{n_1=1}^{N}\frac{\binom{i+b_1-2}{b_1-1}}{n_1^{i+b_1-1}}Z_{n_1-1}(A_i'\sqcup\!\sqcup B_2)+\sum_{i=1}^{b_1}\sum_{n_1=1}^{N}\frac{\binom{i+a_1-2}{a_1-1}}{n_1^{i+a_1-1}}Z_{n_1-1}(A_2\sqcup\!\sqcup B_i'). \tag{3.25}$$

By Lemma 3.3, using the notations for A_i', B_i', A_2, B_2 of Lemma 3.2, we get

$$\sum_{i=1}^{a_1}\sum_{n_1=1}^{N}\frac{\binom{i+b_1-2}{b_1-1}}{n_1^{i+b_1-1}}Z_{n_1-1}(A_i'\sqcup\!\sqcup B_2)+\sum_{i=1}^{b_1}\sum_{n_1=1}^{N}\frac{\binom{i+a_1-2}{a_1-1}}{n_1^{i+a_1-1}}Z_{n_1-1}(A_2\sqcup\!\sqcup B_i')=Z_N(A\sqcup\!\sqcup B). \tag{3.26}$$

Consequently,

$$R_N(\mathbf{a};\mathbf{b})=Z_N(A\sqcup\!\sqcup B). \tag{3.27}$$

This proves the stated result for $R_N(\mathbf{a};\mathbf{b})$ and the corresponding statement of Theorem 2.1. □

Corollary 2.3 can easily be deduced by noting that the sum of the left hand sides of Corollary 2.3 and Theorem 2.1 adds up to $R_N(\mathbf{a};\mathbf{b})$ plus the additional two extra terms. The proof of Corollary 2.4 will be given in the next section, which consists of several remarks.

4. Remarks on Polylogarithms and the Finite Shuffle Identity

For given $\mathbf{a} = (a_1, \ldots, a_r)$ and $\mathbf{b} = (b_1, \ldots, b_s)$, one may define the shuffle product $\zeta_N(\mathbf{a} ⧢ \mathbf{b})$ in terms of the images of the maps Z_N using the words $A = A(\mathbf{a})$ and $B = A(\mathbf{b})$ associated to $\mathbf{a}$ and $\mathbf{b}$ by $A := \omega_0^{a_1-1}\omega_1 \cdots \omega_0^{a_r-1}\omega_1$ and $B := \omega_0^{b_1-1}\omega_1 \cdots \omega_0^{b_s-1}\omega_1$,

$$\zeta_N(\mathbf{a} ⧢ \mathbf{b}) := Z_N(A ⧢ B). \tag{4.1}$$

It turns out that this definition coincides with the usual definition of the shuffle product for multiple zeta values; for an excellent overview concerning the shuffle product for multiple zeta values, we refer the reader to [5, 16, 17].

Let $\mathrm{Li}_{\mathbf{a}}(z)$ denote the (multiple) polylogarithm function with parameters $a_1, \ldots, a_r$, defined by

$$\mathrm{Li}_{\mathbf{a}}(z) = \mathrm{Li}_{a_1,\ldots,a_r}(z) = \sum_{n_1 > n_2 > \cdots > n_r \geq 1} \frac{z^{n_1}}{n_1^{a_1} n_2^{a_2} \cdots n_r^{a_r}}. \tag{4.2}$$

The value $R_N(\mathbf{a};\mathbf{b})$ can be obtained by coefficient extraction in the following way:

$$R_N(\mathbf{a};\mathbf{b}) = \sum_{k=1}^{N} \frac{\zeta_{k-1}(a_2, \ldots, a_r)\zeta_{N-k}(b_1, \ldots, b_s)}{k^{a_1}} = \left[z^N\right] \frac{\mathrm{Li}_{\mathbf{a}}(z)\mathrm{Li}_{\mathbf{b}}(z)}{1-z}. \tag{4.3}$$

On the other hand, by the finite shuffle identity (3.23) for $R_N(\mathbf{a};\mathbf{b})$, one can show the following representation:

$$R_N(\mathbf{a};\mathbf{b}) = \left[z^N\right] \frac{\mathrm{Li}_{\mathbf{a} ⧢ \mathbf{b}}(z)}{1-z}. \tag{4.4}$$

Here the shuffle product for polylogarithm functions $\mathrm{Li}_{\mathbf{a} ⧢ \mathbf{b}}(z)$ is defined in the usual way. We do not want to go into the proof details concerning the equation above since we would have to state and use the precise definition of the shuffle product for multiple zeta values and polylogarithm functions; avoiding repetition, we skip the details and only refer the interested reader to [17], and Theorem 5.1. We want to remark that the result of Proposition 3.4 for $R_N(\mathbf{a};\mathbf{b})$ implies that the shuffle identity for polylogarithm functions, and consequently also for multiple zeta values, can be developed entirely from finite sums using only basic partial fraction decomposition and the combinatorics behind the shuffle product and the shuffle algebra; see Hoffman [13] for an important discussion of the shuffle product. Note that by evaluating at $z = 1$, the shuffle identity for polylogarithm functions implies the shuffle identity for multiple zeta values. The identity above is well known; see for example the article [5]. The shuffle identity for polylogarithm functions is due to the iterated Drinfeld integral representation of polylogarithm functions and multiple zeta values due to Kontsevich [9]. As remarked in [5], the shuffle identity for polylogarithm functions can be deduced from the fact that the product of two simplex integrals consists of a sum of simplex integrals over all possible interlacings of the respective variables of integration.

Finally, we turn to the proof of Corollary 2.4. For $N = 2n+1$ and $j = n+1$ and $n \to \infty$, we have

$$\lim_{n\to\infty} \zeta_j(\mathbf{a})\zeta_{N+1-j}(\mathbf{b}) = \lim_{n\to\infty} \zeta_{n+1}(\mathbf{a})\zeta_{n+1}(\mathbf{b}) = \zeta(\mathbf{a})\zeta(\mathbf{b}),$$
$$\lim_{n\to\infty} \frac{\zeta_n(b_2,\ldots,b_s)\zeta_n(a_2,\ldots,a_r)}{(n+1)^{a_1+b_1}} = 0, \tag{4.5}$$
$$\lim_{n\to\infty} R_N(\mathbf{a};\mathbf{b}) = \lim_{n\to\infty} \zeta_{2n+1}(\mathbf{a}\sqcup\!\sqcup\mathbf{b}) = \zeta(\mathbf{a})\zeta(\mathbf{b}),$$

and the stated result follows.

5. The Reciprocity Relation for Weighted Multiple Zeta Values

Results similar to Theorem 2.1 and Corollary 2.4 can be obtained for products of weighted finite multiple zeta values, $\zeta_N(a_1, a_2, \ldots, a_r; \sigma_1, \ldots, \sigma_r)$, $\sigma_i \in \mathbb{R} \setminus \{0\}$ for $1 \le i \le r$, defined as follows:

$$\zeta_N(\mathbf{a}, \boldsymbol{\sigma}) = \zeta_N(a_1, a_2, \ldots, a_r; \sigma_1, \ldots, \sigma_r) = \sum_{N \ge n_1 > n_2 > \cdots > n_r \ge 1} \frac{1}{\prod_{i=1}^{r} n_i^{a_i} \sigma_i^{n_i}}. \tag{5.1}$$

Of particular interest are the cases $\sigma_i \in \{\pm 1\}$ corresponding to a mixture of alternating and nonalternating signs, which are of particular importance in particle physics. We only state the result generalizing Theorem 2.1, with respect to the notations $\mathbf{a}_2 = (a_2, \ldots, a_r)$, $\boldsymbol{\sigma}_2 = (\sigma_2, \ldots, \sigma_r)$, and the corresponding notations for $\mathbf{b}_2$ and $\boldsymbol{\tau}_2$, and leave the generalizations of Corollaries 2.3 and 2.4 to the reader.

Theorem 5.1. *The multiple zeta values $\zeta_N(\mathbf{a}, \boldsymbol{\sigma})$ and $\zeta_N(\mathbf{b}, \boldsymbol{\tau})$ with weights $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ satisfy the following reciprocity relation:*

$$\begin{aligned} &\sum_{k=1}^{j} \frac{\zeta_{k-1}(\mathbf{b}_2, \boldsymbol{\tau}_2)\zeta_{N-k}(\mathbf{a}, \boldsymbol{\sigma})}{k^{b_1}\tau_1^k} + \sum_{k=1}^{N+1-j} \frac{\zeta_{k-1}(\mathbf{a}_2, \boldsymbol{\sigma}_2)\zeta_{N-k}(\mathbf{b}, \boldsymbol{\tau})}{k^{a_1}\sigma_1^k} \\ &\quad = \zeta_{N+1-j}(\mathbf{a}, \boldsymbol{\sigma})\zeta_j(\mathbf{b}, \boldsymbol{\tau}) - \frac{\zeta_{j-1}(\mathbf{b}_2, \boldsymbol{\tau}_2)\zeta_{N-j}(\mathbf{a}_2, \boldsymbol{\sigma}_2)}{\tau_1^j j^{b_1} \sigma_1^{N+1-j}(N+1-j)^{a_1}} + R_N(\mathbf{a}, \boldsymbol{\sigma}; \mathbf{b}, \boldsymbol{\tau}). \end{aligned} \tag{5.2}$$

Here $R_N(\mathbf{a}, \boldsymbol{\sigma}; \mathbf{b}, \boldsymbol{\tau}) = \sum_{k=1}^{N} \zeta_{N-k}(\mathbf{b}, \boldsymbol{\tau})\zeta_{k-1}(\mathbf{a}_2, \boldsymbol{\sigma}_2)/\sigma_1^k k^{a_1} = R_N(\mathbf{b}, \boldsymbol{\tau}; \mathbf{a}, \boldsymbol{\sigma})$ satisfies an analogue of the shuffle identity with respect to the weights $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$.

The proof of Theorem 2.1 can easily be adapted to the weighted case. Hence, we only elaborate on the main new difficulty, namely, the evaluation of the quantity

$$R_N(\mathbf{a}, \boldsymbol{\sigma}; \mathbf{b}, \boldsymbol{\tau}) = \sum_{k=1}^{N} \frac{\zeta_{N-k}(\mathbf{b}, \boldsymbol{\tau})\zeta_{k-1}(\mathbf{a}_2; \boldsymbol{\sigma}_2)}{\sigma_1^k k^{a_1}}. \tag{5.3}$$

Proceeding as before, that is, taking differences and using partial fraction decomposition, we obtain the recurrence relation

$$\begin{aligned} R_N(\mathbf{a},\boldsymbol{\sigma};\mathbf{b},\boldsymbol{\tau}) = {} & \sum_{i=1}^{a_1}\sum_{n_1=1}^{N}\frac{\binom{i+b_1-2}{b_1-1}}{n_1^{i+b_1-1}\tau_1^{n_1}}R_{n_1-1}\left(a_1+1-i,\mathbf{a}_2,\frac{\tau_1}{\sigma_1},\boldsymbol{\sigma}_2;\mathbf{b}_2,\boldsymbol{\tau}_2\right) \\ & +\sum_{i=1}^{b_1}\sum_{n_1=1}^{N}\frac{\binom{i+a_1-2}{a_1-1}}{n_1^{i+a_1-1}\sigma_1^{n_1}}R_{n_1-1}\left(\mathbf{a}_2,\boldsymbol{\sigma}_2;b_1+1-i,\mathbf{b}_2,\frac{\sigma_1}{\tau_1},\boldsymbol{\tau}_2\right). \end{aligned} \tag{5.4}$$

Consequently, the value $R_N(\mathbf{a},\boldsymbol{\sigma};\mathbf{b},\boldsymbol{\tau})$ can be evaluated into sums of weighted finite multiple zeta values according to a shuffle identity with respect to the weights $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$. We omit the precise definition of this generalization and leave the details to the interested reader.

6. Conclusion

We presented a reciprocity relation for finite multiple zeta values, extending the previous results of [1, 3]. The reciprocity relation involves a shuffle product identity for (finite) multiple zeta values, for which we gave a proof using only partial fraction decomposition and the combinatorial properties of the shuffle product. Moreover, we also presented the reciprocity relation for weighted finite multiple zeta values.

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