IDENTITIES INVOLVING RATIONAL SUMS BY INVERSION AND PARTIAL FRACTION DECOMPOSITION

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Abstract. Identities appearing recently in [2] are treated by inverting them; the resulting sums are evaluated using partial fraction decomposition, following Wenchang Chu [1]. This approach produces a general formula, not only special cases.

1. Introduction

In [2], we find the sums

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\frac{1}{\binom{x+k}{k}}\sum_{1\le i\le j\le k}\frac{1}{x^2+(i+j)x+ij}=\frac{n}{(x+n)^3},$$

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\mathsf{complicated}(k)=\frac{n}{(x+n)^4}.$$

Here, we will discuss an alternative approach to such identities, which will produce the general formula.

It is based on two principles: *inverse pairs* and *partial fraction decomposition.*

I am sure that many other approaches will also work, but I have chosen one that I find useful and appealing. Of course, it is not limited to the sums treated in this paper.

2. Inverse pairs

The following inverse relations are well-known:

$$b_n=\sum_{k=0}^{n}\binom{n}{k}(-1)^{k-1}a_k \longleftrightarrow a_n=\sum_{k=0}^{n}\binom{n}{k}(-1)^{k-1}b_k.$$

They are also easy to prove, e.g., with the use of exponential generating functions.

So, if we want a "nice" answer, like $b_n=\frac{n}{(x+n)^2}$, we must compute

$$a_n=\sum_{k=0}^{n}\binom{n}{k}(-1)^{k-1}b_k$$

to find the "complicated" term.

A technical comment: We will treat $x=0$ as a limiting case, otherwise we would have trouble with b_0, and we would have to artificially define it as 0.

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The computation of

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} \frac{k}{(x+k)^2}$$

(and similar quantities) will be treated in the next section.

3. Partial fraction decomposition

The following approach is based on [1].

Consider (for $n \ge 1$)

$$T := \frac{n!}{z(z-1)\dots(z-n)} \frac{z}{(x+z)^2}$$

and perform partial fraction decomposition:

$$T = \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} \frac{k}{(x+k)^2} \frac{1}{z-k} + \frac{\lambda}{(x+z)^2} + \frac{\mu}{x+z}.$$

Now we multiply this relation by z and let $z \to \infty$ to find

$$0 = \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} \frac{k}{(x+k)^2} + \mu.$$

This evalutes the sum:

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \frac{k}{(x+k)^2} = (-1)^n \mu.$$

Now

$$\begin{aligned}
(-1)^n \mu &= (-1)^n [(x+z)^{-1}] \frac{n!}{z(z-1)\dots(z-n)} \frac{z}{(x+z)^2} \\
&= (-1)^n [(x+z)^1] \frac{n!}{(z-1)\dots(z-n)} \\
&= [z^1] \frac{n!}{(1+x-z)\dots(n+x-z)} \\
&= \frac{n!}{(1+x)\dots(n+x)} \sum_{k=1}^{n} \frac{1}{k+x}.
\end{aligned}$$

This produces the identity

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \frac{1}{\binom{k+x}{x}} \sum_{j=1}^{k} \frac{1}{j+x} = \frac{n}{(x+n)^2}.$$

This instance was the warm-up for the general instance $b_n = \frac{n}{(x+n)^{d+1}}$, which is not much more complicated.

Analogous computations lead to

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\frac{k}{(x+k)^{d+1}}=(-1)^n\mu$$

with

$$\begin{aligned}(-1)^n\mu &= (-1)^n[(x+z)^{-1}]\frac{n!}{z(z-1)\dots(z-n)}\frac{z}{(x+z)^{d+1}}\\ &= (-1)^n[(x+z)^d]\frac{n!}{(z-1)\dots(z-n)}\\ &= [z^d]\frac{n!}{(1+x-z)\dots(n+x-z)}\\ &= \frac{n!}{(1+x)\dots(n+x)}[z^d]\frac{1}{(1-\frac{z}{1+x})\dots(1-\frac{z}{n+x})}\\ &= \frac{1}{\binom{x+n}{n}}[z^d]\exp\left(\log\frac{1}{1-\frac{z}{1+x}}+\dots+\log\frac{1}{1-\frac{z}{n+x}}\right)\\ &= \frac{1}{\binom{x+n}{n}}[z^d]\exp\left(\sum_{k=1}^{n}\sum_{j\geq 1}\frac{z^j}{j(k+x)^j}\right)\\ &= \frac{1}{\binom{x+n}{n}}[z^d]\exp\left(\sum_{j\geq 1}\frac{s_{n,j}z^j}{j}\right)\\ &= \frac{1}{\binom{x+n}{n}}[z^d]\prod_{j\geq 1}\sum_{l\geq 0}\frac{s_{n,j}^l z^{jl}}{l!j^l},\end{aligned}$$

with

$$s_{n,j}=\sum_{k=1}^{n}\frac{1}{(k+x)^j}.$$

Consequently

$$(-1)^n\mu=\frac{1}{\binom{x+n}{n}}\sum_{l_1+2l_2+3l_3+\dots=d}\frac{s_{n,1}^{l_1}s_{n,2}^{l_2}\dots}{l_1!l_2!\dots 1^{l_1}2^{l_2}\dots}.$$

Theorem 1.

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\frac{k}{(x+k)^{d+1}}=\frac{1}{\binom{x+n}{n}}\sum_{l_1+2l_2+3l_3+\dots=d}\frac{s_{n,1}^{l_1}s_{n,2}^{l_2}\dots}{l_1!l_2!\dots 1^{l_1}2^{l_2}\dots}$$

with

$$s_{n,j}=\sum_{k=1}^{n}\frac{1}{(k+x)^j}.$$

For instance, we recover the formula for $d=2$, since

$$\frac{s_{n,1}^2+s_{n,2}}{2}=\sum_{1\le i\le j\le n}\frac{1}{x^2+(i+j)x+ij},$$

as one can easily check. The other instance $d=3$, given in [2] evaluates here handily as $s_{n,1}^3/6+s_{n,1}s_{n,2}/2+s_{n,3}/3$.

References

[1] Wenchang Chu. A binomial coefficient identity associated with Beukers' conjecture on Apéry numbers. *Electron. J. Combin.*, 11(1):Note 15, 3 pp. (electronic), 2004.

[2] J.L. Díaz-Barrero, J. Gibergans-Báguena, and P.G. Popescu. Some identities involving rational sums. *Applicable Analysis and Discrete Mathematics*, 1, 397–402, 2007.

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