A CONTINUED FRACTION EXPANSION FOR A q-TANGENT FUNCTION: AN ELEMENTARY PROOF

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ABSTRACT. We prove a continued fraction expansion for a certain q-tangent function that was conjectured by the present writer, then proved by Fulmek, now in a completely elementary way.

1. INTRODUCTION

In [3], the present writer defined the following q-trigonometric functions

$$\sin_q(z) = \sum_{n \geq 0} \frac{(-1)^n z^{2n+1}}{[2n+1]_q!} q^{n^2},$$

$$\cos_q(z) = \sum_{n \geq 0} \frac{(-1)^n z^{2n}}{[2n]_q!} q^{n^2}.$$

Here, we use standard q-notation:

$$[n]_q := \frac{1-q^n}{1-q}, \qquad [n]_q! := [1]_q [2]_q \ldots [n]_q.$$

These q-functions are variations of Jackson's [2] q-sine and q-cosine functions.

For the q-tangent function $\tan_q(z) = \frac{\sin_q(z)}{\cos_q(z)}$, the following continued fraction expansion was conjectured in [3]:

$$z \tan_q(z) = \cfrac{z^2}{[1]_q q^0 - \cfrac{z^2}{[3]_q q^{-2} - \cfrac{z^2}{[5]_q q^1 - \cfrac{z^2}{[7]_q q^{-9} - \cdots}}}}.$$

Here, the powers of q are of the form $(-1)^{n-1} n(n-1)/2 - n + 1$.

In [1], this statement was proven using heavy machinery from q-analysis.

Happily, after about 8 years, I was now successful to provide a complete *elementary* proof that I will present in the next section.

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2. THE PROOF

We write

$$\frac{z\sin_q(z)}{\cos_q(z)} = \frac{z^2}{N_0} = \frac{z^2}{C_1 - \dfrac{z^2}{N_1}} = \frac{z^2}{C_1 - \dfrac{z^2}{C_2 - \dfrac{z^2}{N_2}}} = \cdots,$$

and set

$$N_i = \frac{a_i}{b_i}.$$

This means

$$N_i = C_{i+1} - \frac{z^2}{N_{i+1}}$$

or

$$\frac{z^2}{N_{i+1}} = C_{i+1} - N_i$$

and

$$\frac{b_{i+1}z^2}{a_{i+1}} = C_{i+1} - \frac{a_i}{b_i} = \frac{C_{i+1}b_i - a_i}{b_i}.$$

Therefore $a_i = b_{i-1}$ and

$$b_{i+1}z^2 = C_{i+1}b_i - b_{i-1}.$$

The initial conditions are

$$b_{-1} = \cos_q(z) \quad \text{and} \quad b_0 = \sum_{n\geq 0} \frac{(-1)^n q^{n^2} z^{2n}}{[2n+1]_q!}.$$

The constants C_i guarantee that all the b_i are power series, i.e., they make the constant term in $C_{i+1}b_i - b_{i-1}$ disappear. Our goal is to show that $C_i = [2i-1]_q q^{(-1)^{i-1}i(i-1)/2-i+1}$ are the (unique) numbers that do this. We are proving the claim by proving the following *explicit* formula for b_i:

$$b_i = \sum_{n\geq 0} \frac{(-1)^n z^{2n}}{[2n+2i+1]_q!} \left(\prod_{j=1}^{i}[2n+2j]_q\right) q^{(n+\lfloor\frac{i+1}{2}\rfloor)^2 + [i\text{ odd}]\binom{i+1}{2}}.$$

Note that the C_i are uniquely determined by the imposed condition, and since the b_i are power series, we are done once we prove this formula by induction. The first two instances satisfy this, and we do the induction step now:

$$C_{i+1}b_i - b_{i-1} = [2i+1]_q q^{(-1)^i\binom{i+1}{2}-i} \sum_{n\geq 0} \frac{(-1)^n z^{2n}}{[2n+2i+1]_q!} \left(\prod_{j=1}^{i}[2n+2j]_q\right) q^{(n+\lfloor\frac{i+1}{2}\rfloor)^2 + [i\text{ odd}]\binom{i+1}{2}}$$

$$- \sum_{n\geq 0} \frac{(-1)^n z^{2n}}{[2n+2i-1]_q!} \left(\prod_{j=1}^{i-1}[2n+2j]_q\right) q^{(n+\lfloor\frac{i}{2}\rfloor)^2 + [i-1\text{ odd}]\binom{i}{2}}$$

$$= \sum_{n\geq 0} \frac{(-1)^n z^{2n}}{[2n+2i+1]_q!} \left([2i+1]_q\left(\prod_{j=1}^{i}[2n+2j]_q\right) q^{(n+\lfloor\frac{i+1}{2}\rfloor)^2 + [i\text{ odd}]\binom{i+1}{2}+(-1)^i\binom{i+1}{2}-i}\right.$$

$$- [2n + 2i + 1]_q \left(\prod_{j=1}^{i} [2n + 2j]_q \right) q^{(n + \lfloor \frac{i}{2} \rfloor)^2 + [i-1 \text{ odd}]\binom{i}{2}} \Bigg)$$

$$= \frac{1}{1-q} \sum_{n \geq 0} \frac{(-1)^n z^{2n}}{[2n + 2i + 1]_q!} \left(\prod_{j=1}^{i} [2n + 2j]_q \right) \times$$

$$\times \left((1 - q^{2i+1}) q^{(n + \lfloor \frac{i+1}{2} \rfloor)^2 + [i \text{ even}]\binom{i+1}{2} - i} - (1 - q^{2n+2i+1}) q^{(n + \lfloor \frac{i}{2} \rfloor)^2 + [i \text{ even}]\binom{i}{2}} \right).$$

The last bracket in this expression can be simplified for i even:

$$-q^{(n + \frac{i}{2})^2 + \binom{i}{2} + 2i + 1}(1 - q^{2n})$$

and for i odd:

$$-q^{(n + \frac{i-1}{2})^2}(1 - q^{2n}).$$

Putting everything together, we arrive at

$$C_{i+1} b_i - b_{i-1} = \sum_{n \geq 0} \frac{(-1)^{n-1} z^{2n}}{[2n + 2i + 1]_q!} \left(\prod_{j=0}^{i} [2n + 2j]_q \right) q^{(n + \lfloor \frac{i}{2} \rfloor)^2 + [i \text{ even}]\binom{i+2}{2}}.$$

Notice that the constant term vanishes, whence

$$b_{i+1} = \sum_{n \geq 1} \frac{(-1)^{n-1} z^{2n-2}}{[2n + 2i + 1]_q!} \left(\prod_{j=0}^{i} [2n + 2j]_q \right) q^{(n + \lfloor \frac{i}{2} \rfloor)^2 + [i \text{ even}]\binom{i+2}{2}}$$

$$= \sum_{n \geq 0} \frac{(-1)^n z^{2n}}{[2n + 2(i+1) + 1]_q!} \left(\prod_{j=1}^{i+1} [2n + 2j]_q \right) q^{(n + \lfloor \frac{i+2}{2} \rfloor)^2 + [i + 1 \text{ odd}]\binom{i+2}{2}},$$

which is the announced formula.

References

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