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Infinitely many Identities for Sums of Jacobsthal-Lucas Numbers

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1. Generation of Identities

The numbers in the title are defined as $j(n) = 2^n + (-1)^n$. In [1] we find 18 theorems evaluating sums of the type

$$\sum_{k=m}^{n} \varepsilon^k j(a_1k + b_1) \dots j(a_sk + b_s),$$

where $\varepsilon \in \{1, -1\}$, $a_i \in \mathbb{N}$, $b_i \in \mathbb{Z}$.

Here I will describe how to obtain as many theorems as one wishes, within a few minutes, preferably assisted by a computer.

It is enough to consider a sum $\sum_{0\le k<n} \dots$, as the more general instance is then just the difference of 2 formulæ. Using the explicit form of the Jacobsthal-Lucas numbers, a typical summand is of the form

$$c_0 + c_1 2^k + \dots + c_s 2^{sk} + (-1)^k[d_0 + d_1 2^k + \dots + d_s 2^{sk}].$$

Performing the summation, which is just the application of the formula for the finite geometric series (!), the evaluation of

$$\sum_{k=0}^{n-1} \varepsilon^k j(a_1k + b_1) \dots j(a_sk + b_s)$$

is given by

$$e_0 + e_1 2^n + \dots + e_s 2^{sn} + (-1)^n[f_0 + f_1 2^n + \dots + f_s 2^{sn}] + Bn.$$

The computation of the constants is preferably done by a computer.

If one wants to express the answer in terms of Jacobsthal-Lucas numbers, one performs the substitution $2^{in} \longrightarrow (j(n)-(-1)^n)^i$ and asks for simplification.

To illustrate, I produce some random theorems:

Example 1.

$$\begin{aligned}\sum_{k=0}^{n-1} j(2k)j(3k+1)j(5k+3) &= \tfrac{16}{1023}j(n)^{10} + \tfrac{66656}{86955}j(n)^8 + \tfrac{17226664}{3739065}j(n)^6\\ &+ \tfrac{2984462}{747813}j(n)^4 - \tfrac{8245711}{3739065}j(n)^2 + n - \tfrac{5000}{4257}\\ &+ (-1)^n\big[-\tfrac{160}{1023}j(n)^9 - \tfrac{2887528}{1246355}j(n)^7 - \tfrac{7290826}{1246355}j(n)^5 + \tfrac{364666}{11217195}j(n)^3 + \tfrac{4826746}{3739065}j(n)\big].\end{aligned}$$

Example 2.

$$\begin{aligned}\sum_{k=0}^{n-1} j(5k)^3 j(7k+12)(-1)^k &= \tfrac{90112}{4194305}j(n)^{21} + \tfrac{1261568}{838861}j(n)^{19} + \tfrac{14189390352384}{549751750655}j(n)^{17}\\ &+ \tfrac{3229804503436591103}{18013715613712385}j(n)^{15} + \tfrac{3648505730491 67869}{514677588963211}j(n)^{13} + \tfrac{1987587938472868 5809469}{10543170409911377335}j(n)^{11}\\ &+ \tfrac{1480465229166713661733567}{43226998680636 6470735}j(n)^9 + \tfrac{169011038047689459007191 1283}{384288018270859792483415}j(n)^7\\ &+ \tfrac{1004496929251786982250800334}{2744914416220427 08916725}j(n)^5 + \tfrac{1018497773265946728246208 83}{548982883244085417 83345}j(n)^3\\ &+ \tfrac{1014073964584357688357480 64}{384288018270859792483415}j(n) - \tfrac{1112104653095323142808832 67}{3842880182708597924834150}\\ &- (-1)^n\Big[\tfrac{4096}{4194305}j(n)^{22} + \tfrac{946176}{4194305}j(n)^{20} + \tfrac{5992448}{838861}j(n)^{18}\\ &+ \tfrac{40933417279488}{549751750655}j(n)^{16} + \tfrac{135472602336733 5933}{3602743122742477}j(n)^{14} + \tfrac{12806079157051973967807}{10543170409911377335}j(n)^{12}\\ &+ \tfrac{573362962390529553 4880536}{2161349934031832353675}j(n)^{10} + \tfrac{1233449808330260032645 6068}{30258899076445652951 45}j(n)^8\\ &+ \tfrac{232273431401329810610991001}{548982883244085417 83345}j(n)^6 + \tfrac{156178320877049624927144594}{548982883244085417 83345}j(n)^4\\ &+ \tfrac{488222197422355249364142 72}{548982883244085417 83345}j(n)^2 + \tfrac{138134999275414423270200333}{3842880182708597924834150}\Big].\end{aligned}$$

It is apparent that these comments apply *mutatis mutandis* to other sequences of numbers, like Jacobsthal numbers, Pell numbers, etc.

This paper could safely end here. But I was asked by the editors to write 4 pages, so I'll add some more material.

Summation as in this article is best understood via generating functions: If

$$f(z) = \sum_{n\ge0} a_n z^n,$$

then

$$\frac{1}{1-z}f(z) = \sum_{n\ge0}(a_0+a_1+\cdots+a_n)z^n,$$

and we find a desired sum $a_0+a_1+\cdots+a_n$ as the coefficient of z^n in $\frac{1}{1-z}f(z)$.

In other words, on the level of generating functions, summing means just multiplication with $1/(1-z)$. Since eventually a term $1/(1-z)^2$ might appear after this multiplication, we expect in our answer terms of the form *constant*$\cdot n$.

2. Identities for Jacobsthal numbers

Let us do one example for the Jacobsthal numbers $J(n) = \frac{2^n-(-1)^n}{3}$.

$$\begin{aligned}\sum_{k=0}^{n-1} J(3k)^5 &= \tfrac{1}{114251464467567}J(n)^{15} + \tfrac{5}{5440545927027}J(n)^{13} - \tfrac{2261435}{22289916663029619}J(n)^{11}\\ &+ \tfrac{70262803235}{34170442244424405927}J(n)^{9} + \tfrac{75861304565}{542387972133720729}J(n)^{7} - \tfrac{99477394916093}{21153130913215108431}J(n)^{5}\\ &+ \tfrac{40797090205508200}{44421574917751727051}J(n)^{3} + \tfrac{47594930628766216}{148071916392505759017}J(n) - \tfrac{1456397415}{254799665302}\\ &+ (-1)^n\Big[\tfrac{5}{38083821489189}J(n)^{14} + \tfrac{898240}{66869749989088857}J(n)^{12}\\ &\qquad + \tfrac{14488441}{22289916663029619}J(n)^{10} + \tfrac{157782411620}{379671580493604510 3}J(n)^{8}\\ &\qquad + \tfrac{77479289741561}{634593927396453252 93}J(n)^{6} + \tfrac{286565869066390}{21153130913215108431}J(n)^{4}\\ &\qquad + \tfrac{50311901484804520}{148071916392505759017}J(n)^{2} + \tfrac{19255457014722338 47}{8884314983550345541 02}\Big].\end{aligned}$$

3. Identities for Pell numbers

Let us consider the Pell numbers

$$P(n) = \frac{\alpha^n - \beta^n}{2\sqrt{2}}, \quad \text{with} \quad \alpha = 1 + \sqrt{2} \quad \text{and} \quad \beta = 1 - \sqrt{2}.$$

The associated sequence is ("Pell-Lucas numbers")

$$Q(n) = \alpha^n + \beta^n,$$

so that

$$\alpha^n = \frac{Q(n) + 2\sqrt{2}P(n)}{2}, \qquad \beta^n = \frac{Q(n) - 2\sqrt{2}P(n)}{2}.$$

With these formulæ, we can express the final formula of a summation in terms of $P(n)$ and $Q(n)$.

Let us do an example:

$$S(n) = \sum_{k=0}^{n-1} P(n)^3.$$

We find the generating function (which we present already in partial fractioned form):

$$\begin{aligned}\sum_{n\ge 0} S(n)z^n &= \frac{\sqrt{2}}{56(1-z\alpha^3)} + \frac{5}{224(1-z\alpha^3)} - \frac{\sqrt{2}}{56(1-z\beta^3)} + \frac{5}{224(1-z\beta^3)}\\ &- \frac{3}{32(1+z\alpha)} - \frac{3}{32(1+z\beta)}.\end{aligned}$$

From this we conclude

$$\begin{aligned} S(n) &= \frac{\sqrt{2}}{56}\big(\alpha^{3n}-\beta^{3n}\big)+\frac{5}{224}\big(\alpha^{3n}+\beta^{3n}\big)-\frac{3}{32}\big(\alpha^{n}+\beta^{n}\big)(-1)^n+\frac{1}{7}\\ &= \frac{1}{14}P(3n)+\frac{5}{224}Q(3n)-\frac{3}{32}(-1)^nQ(n)+\frac{1}{7}. \end{aligned}$$

And here is one last example:

$$S(n)=\sum_{k=0}^{n-1}Q(k)^4(-1)^k.$$

We compute

$$\begin{aligned} \sum_{n\ge0}S(n)z^n &= \frac{\sqrt{2}}{3}\frac{1}{1+z\alpha^4}+\frac{1}{2(1+z\alpha^4)}-\frac{\sqrt{2}}{3}\frac{1}{1+z\beta^4}+\frac{1}{2(1+z\beta^4)}\\ &\quad+\frac{2\sqrt{2}}{1-z\alpha^2}+\frac{2}{1-z\alpha^2}-\frac{2\sqrt{2}}{1-z\beta^2}+\frac{2}{1-z\beta^2}+\frac{8}{1-z}+\frac{3}{1+z}. \end{aligned}$$

Therefore

$$\begin{aligned} S(n) &= \frac{\sqrt{2}}{3}(-1)^n\big(\alpha^{4n}-\beta^{4n}\big)+\frac{1}{2}(-1)^n\big(\alpha^{4n}+\beta^{4n}\big)\\ &\quad+2\sqrt{2}\big(\alpha^{2n}-\beta^{2n}\big)+2\big(\alpha^{2n}+\beta^{2n}\big)+8+3(-1)^n\\ &= \frac{4}{3}(-1)^nP(4n)+\frac{1}{2}(-1)^nQ(4n)+8P(2n)+2Q(2n)+8+3(-1)^n. \end{aligned}$$

References

[1] Z. Čerin, Formulæ for Sums of Jacobsthal-Lucas Numbers, *International Mathematical Forum,* **2** (2007), 1969–1984.

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