A GENERALIZED FILBERT MATRIX

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Abstract. A generalized Filbert matrix is introduced, sharing properties of the Hilbert matrix and Fibonacci numbers. Explicit formulæ are derived for the LU-decomposition, their inverses, and the Cholesky factorization. The approach is to use q-analysis and to leave the justification of the necessary identities to the q-version of Zeilberger's celebrated algorithm.

1. Introduction

The Filbert matrix $H_n = (h_{ij})_{i,j=1}^{n}$ is defined by $h_{ij} = \frac{1}{F_{i+j-1}}$ as an analogue of the Hilbert matrix where F_n is the nth Fibonacci number. It has been defined and studied by Richardson [1].

In this paper we will study the generalized matrix with entries $\frac{1}{F_{i+j+r}}$, where $r \geq -1$ is an integer parameter. The size of the matrix does not really matter, and we can think about an infinite matrix $\mathcal{F}$ and restrict it whenever necessary to the first n rows resp. columns and write $\mathcal{F}_n$.

Our approach will be as follows. We will use the Binet form

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q},$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$. All the identities we are going to derive hold for general q, and results about Fibonacci numbers come out as corollaries for the special choice of q.

Throughout this paper we will use the following notations: $(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1})$ and the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

Furthermore, we will use *Fibonomial coefficients*

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{F_n F_{n-1} \dots F_{n-k+1}}{F_1 \dots F_k}.$$

The link between the two notations is

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \alpha^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix} \quad \text{with} \quad q = -\alpha^{-2}.$$

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We will obtain the LU-decomposition $\mathcal{F} = L \cdot U$:

Theorem 1. *For $1 \le d \le n$ we have*

$$L_{n,d} = q^{\frac{n-d}{2}} \mathbf{i}^{n+d} (-1)^n \begin{bmatrix} n-1 \\ d-1 \end{bmatrix} \begin{bmatrix} 2d+r \\ d \end{bmatrix} \begin{bmatrix} n+d+r \\ d \end{bmatrix}^{-1}$$

and its Fibonacci corollary

$$L_{n,d} = \left\{ \begin{matrix} n-1 \\ d-1 \end{matrix} \right\} \left\{ \begin{matrix} 2d+r \\ d \end{matrix} \right\} \left\{ \begin{matrix} n+d+r \\ d \end{matrix} \right\}^{-1}.$$

Theorem 2. *For $1 \le d \le n$ we have*

$$U_{d,n} = q^{\frac{n-d-r-1}{2}+d^2+rd} \mathbf{i}^{n+d+r+1} (-1)^{n+d+r} \begin{bmatrix} 2d+r-1 \\ d-1 \end{bmatrix}^{-1} \begin{bmatrix} n+d+r \\ d \end{bmatrix}^{-1} \begin{bmatrix} n \\ d \end{bmatrix} \frac{1-q}{1-q^n}$$

and its Fibonacci corollary

$$U_{d,n} = (-1)^{r(d+1)} \left\{ \begin{matrix} 2d+r-1 \\ d-1 \end{matrix} \right\}^{-1} \left\{ \begin{matrix} n+d+r \\ d \end{matrix} \right\}^{-1} \left\{ \begin{matrix} n \\ d \end{matrix} \right\} \frac{1}{F_n}.$$

We could also determine the inverses of the matrices L and U:

Theorem 3. *For $1 \le d \le n$ we have*

$$L^{-1}_{n,d} = q^{\frac{(n-d)^2}{2}} \mathbf{i}^{n+d} (-1)^d \begin{bmatrix} n+r \\ d+r \end{bmatrix} \begin{bmatrix} n+d-1+r \\ d-1 \end{bmatrix} \begin{bmatrix} 2n-1+r \\ n-1 \end{bmatrix}^{-1}$$

and its Fibonacci corollary

$$L^{-1}_{n,d} = (-1)^{(n+1)d+\frac{n(n+1)}{2}+\frac{d(d+1)}{2}} \left\{ \begin{matrix} n+r \\ d+r \end{matrix} \right\} \left\{ \begin{matrix} n+d-1+r \\ d-1 \end{matrix} \right\} \left\{ \begin{matrix} 2n-1+r \\ n-1 \end{matrix} \right\}^{-1}.$$

Theorem 4. *For $1 \le d \le n$ we have*

$$U^{-1}_{d,n} = q^{-\frac{n^2}{2}+\frac{d^2}{2}+\frac{r+1}{2}-(d+r)n} \mathbf{i}^{n+d-1+r} (-1)^{n-d} \begin{bmatrix} 2n+r \\ n \end{bmatrix} \begin{bmatrix} n+d+r-1 \\ d+r \end{bmatrix} \begin{bmatrix} n-1 \\ d-1 \end{bmatrix} \frac{1-q^n}{1-q}$$

and its Fibonacci corollary

$$U^{-1}_{d,n} = (-1)^{\frac{n(n+1)}{2}+\frac{d(d-1)}{2}-dn-rn+r} \left\{ \begin{matrix} 2n+r \\ n \end{matrix} \right\} \left\{ \begin{matrix} n+d+r-1 \\ d+r \end{matrix} \right\} \left\{ \begin{matrix} n-1 \\ d-1 \end{matrix} \right\} F_n.$$

As a consequence we can compute the determinant of $\mathcal{F}_n$, since it is simply evaluated as $U_{1,1} \cdots U_{n,n}$ (we only state the Fibonacci version):

Theorem 5.

$$\det \mathcal{F}_n = (-1)^{\frac{rn(n-1)}{2}} \prod_{d=1}^{n} \left\{ \begin{matrix} 2d+r-1 \\ d-1 \end{matrix} \right\}^{-1} \left\{ \begin{matrix} 2d+r \\ d \end{matrix} \right\}^{-1} \frac{1}{F_d}.$$

Now we determine the inverse of the matrix $\mathcal{F}$. This time it depends on the dimension, so we compute $(\mathcal{F}_n)^{-1}$.

Theorem 6. *For $1 \le i, j \le n$:*

$$(\mathcal{F}_n)^{-1}_{i,j} = q^{\frac{i^2+j^2+r+1}{2}-(i+j+r)n}\mathbf{i}^{i+j+r+1}(-1)^{i+j+1} \\ \times \begin{bmatrix} n+r+i \\ n \end{bmatrix} \begin{bmatrix} n+r+j \\ n \end{bmatrix} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} \frac{(1-q^n)^2}{(1-q^{r+i+j})(1-q)}$$

and its Fibonacci corollary

$$(\mathcal{F}_n)^{-1}_{i,j} = (-1)^{\frac{i(i-1)}{2}+\frac{j(i-1)}{2}+n(i+j+r)+r} \begin{Bmatrix} n+r+i \\ n \end{Bmatrix} \begin{Bmatrix} n+r+j \\ n \end{Bmatrix} \begin{Bmatrix} n-1 \\ i-1 \end{Bmatrix} \begin{Bmatrix} n-1 \\ j-1 \end{Bmatrix} \frac{F_n^2}{F_{r+i+j}}.$$

We can also find the Cholesky decomposition $\mathcal{F} = \mathcal{C} \cdot \mathcal{C}^T$ with a lower triangular matrix $\mathcal{C}$:

Theorem 7. *For $n \ge d$:*

$$\mathcal{C}_{n,d} = (-1)^n \mathbf{i}^{n+r+\frac{r+1}{2}} q^{\frac{n}{2}-\frac{r+1}{4}+\frac{d(d-1)}{2}+\frac{rd}{2}} \frac{\sqrt{(1-q^{2d+r})(1-q)}}{1-q^{2n+r}} \begin{bmatrix} 2n+r \\ n-d \end{bmatrix} \begin{bmatrix} 2n+r-1 \\ n-1 \end{bmatrix}^{-1}$$

and its Fibonacci corollary

$$\mathcal{C}_{n,d} = (-1)^{\frac{d(d-1)}{2}+\frac{r(d+1)}{2}} \frac{\sqrt{F_{2d+r}}}{F_{2n+r}} \begin{Bmatrix} 2n+r \\ n-d \end{Bmatrix} \begin{Bmatrix} 2n+r-1 \\ n-1 \end{Bmatrix}^{-1}.$$

Notice that for odd r, even the Fibonacci version may contain complex numbers.

2. Proofs

In order to show that indeed $\mathcal{F} = L \cdot U$, we need to show that for any m, n:

$$\sum_d L_{m,d} U_{d,n} = \mathcal{F}_{m,n} = \alpha^{-m-n-r+1} \frac{1-q}{1-q^{m+n+r}}.$$

In rewritten form the formula to be proved reads

$$\sum_d (q^{d^2+(r-1)d-r} - q^{d^2+(r+1)d}) \begin{bmatrix} 2m+r \\ m-d \end{bmatrix} \begin{bmatrix} 2n+r \\ n-d \end{bmatrix} \\ = \frac{(1-q^{2n+r})(1-q^{2m+r})}{1-q^{m+n+r}} \begin{bmatrix} 2m+r-1 \\ m-1 \end{bmatrix} \begin{bmatrix} 2n+r-1 \\ n-1 \end{bmatrix}.$$

Nowadays, such identities are a routine verification using the q-Zeilberger algorithm, as described in the book [2].

For interest, we also state (as a corollary) the corresponding Fibonacci identity:

$$\sum_d (-1)^{r(d-1)} F_{2d+r} \begin{Bmatrix} 2m+r \\ m-d \end{Bmatrix} \begin{Bmatrix} 2n+r \\ n-d \end{Bmatrix} = \frac{F_{2n+r} F_{2m+r}}{F_{m+n+r}} \begin{Bmatrix} 2m+r-1 \\ m-1 \end{Bmatrix} \begin{Bmatrix} 2n+r-1 \\ n-1 \end{Bmatrix}.$$

Now we move to the inverse matrices. Since L and L^{-1} are lower triangular matrices, we only need to look at the entries indexed by (m,n) with $m\ge n$:

$$\begin{aligned}
&\sum_{n\le d\le m} L_{m,d}L^{-1}_{d,n}\\
&=\sum_{n\le d\le m} q^{\frac{m-d}{2}}\mathbf{i}^{m+d}(-1)^m\begin{bmatrix}m-1\\d-1\end{bmatrix}\begin{bmatrix}2d+r\\d\end{bmatrix}\begin{bmatrix}m+d+r\\d\end{bmatrix}^{-1}\\
&\qquad\times q^{\frac{(n-d)^2}{2}}\mathbf{i}^{n+d}(-1)^n\begin{bmatrix}d+r\\n+r\end{bmatrix}\begin{bmatrix}n+d-1+r\\n-1\end{bmatrix}\begin{bmatrix}2d-1+r\\d-1\end{bmatrix}^{-1}\\
&=\frac{1}{1-q^{2m+r}}\begin{bmatrix}2m+r-1\\m-1\end{bmatrix}^{-1}\mathbf{i}^{m+n}(-1)^{m+n}\\
&\qquad\times\sum_{n\le d\le m} q^{\frac{m-d}{2}+\frac{(n-d)^2}{2}}(1-q^{2d+r})(-1)^d\begin{bmatrix}2m+r\\m-d\end{bmatrix}\begin{bmatrix}n+d-1+r\\n+r\end{bmatrix}\begin{bmatrix}d-1\\n-1\end{bmatrix}.
\end{aligned}$$

The q-Zeilberger algorithm can evaluate the sum, and it is indeed $[m=n]$, as predicted.

The argument for $U\cdot U^{-1}$ is similar:

$$\begin{aligned}
&\sum_{m\le d\le n} U_{m,d}U^{-1}_{d,n}\\
&=(-1)^{m+n}\mathbf{i}^{m+n}q^{-\frac{m}{2}+m^2+rm-\frac{n^2}{2}-rn}\begin{bmatrix}2m+r-1\\m-1\end{bmatrix}^{-1}\begin{bmatrix}2n+r-1\\n-1\end{bmatrix}\frac{1-q^{2n+r}}{1-q^{n+m+r}}\\
&\qquad\times\sum_{m\le d\le n}(-1)^d q^{\frac{d(d+1)}{2}-dn}\begin{bmatrix}n+d+r-1\\d-m\end{bmatrix}\begin{bmatrix}n+m+r\\n-d\end{bmatrix}.
\end{aligned}$$

Again, the q-Zeilberger algorithm evaluates this to $[m=n]$.

Now we turn to the inverse matrix:

$$\begin{aligned}
((\mathcal{F}_n)^{-1}\mathcal{F}_n)_{i,k}&=\mathbf{i}^{i-k}(-1)^i q^{\frac{i^2+k}{2}-(i+r)n+r}(1-q^n)^2\begin{bmatrix}n+r+i\\n\end{bmatrix}\begin{bmatrix}n-1\\i-1\end{bmatrix}\\
&\qquad\times\sum_{j=1}^{n} q^{\frac{j(j+1)}{2}-jn}(-1)^j\begin{bmatrix}n+r+j\\n\end{bmatrix}\begin{bmatrix}n-1\\j-1\end{bmatrix}\frac{1}{(1-q^{j+k+r})(1-q^{r+i+j})}.
\end{aligned}$$

And the q-Zeilberger algorithm evaluates this again to $[i=k]$.

The Cholesky verification goes like this:

$$\begin{aligned}
&\sum_{d=1}^{\min\{m,n\}}\mathcal{C}_{m,d}\mathcal{C}_{n,d}\\
&=(-1)^{m+n+r}\mathbf{i}^{m+n+r+1}q^{\frac{m+n-r-1}{2}}\frac{1-q}{(1-q^{2m+r})(1-q^{2n+r})}\begin{bmatrix}2n+r-1\\n-1\end{bmatrix}^{-1}\begin{bmatrix}2m+r-1\\m-1\end{bmatrix}^{-1}\\
&\qquad\times\sum_d q^{d(d-1)+rd}(1-q^{2d+r})\begin{bmatrix}2m+r\\m-d\end{bmatrix}\begin{bmatrix}2n+r\\n-d\end{bmatrix}.
\end{aligned}$$

And again the q-Zeilberger algorithm evaluates this to be

$$\frac{1-q}{1-q^{m+n+r}}\mathbf{i}^{m+n+r-1}q^{\frac{m+n+r-1}{2}},$$

as it should.

References

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