# ADVANCING IN THE PRESENCE OF A DEMON

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Abstract. We study a parameter that contains *approximate counting,* i.e., the level reached after $n$ random increments, driven by geometric probabilities, and *insertion costs* for *tries* as special cases. We are able to compute all moments of this parameter in a semi-automatic fashion. This is another showcase of the machinery developed in an earlier paper of these authors. Roughly speaking, it works when the underlying distributions are distributed according to the *Gumbel* distribution, or something similar.

## 1. Introduction

Assume that $n$ persons want to advance on a staircase. The rules are as follows: The party starts at level 1. The $m$ persons who advanced to level $k$ flip a coin. Those who flip '1' (with probability $q$) advance to the next level; the others, who flipped '0' (with probability $p = 1 - q$) die. Additionally, there is a demon, who kills one of the survivors with probability $\nu$, but lets them alone with probability $\mu = 1 - \nu$. The demon interfers only at a level 2 or higher. If one single person is advancing to level $k$ and is eaten, we *don't* say that this level was reached. Only people who survive the coin flipping *and* the demon count!

As was worked out in [8], the instance $\mu = 0$ corresponds to *approximate counting.* Let us recall what it is, to keep this paper independent. There is a counter (the state the process is in at the moment), starting at 1, and random increments, the increase the counter from $i$ to $i + 1$ with probability $q^i$, otherwise it stays at $i$. One is interested in the value of the counter after $n$ random increments.

The other extremal case $\mu = 1$ (no demon interfering) is related to a digital data structure called *tries* [3, 7]. Although in the previous paper [8], only the symmetrical case $p = q = \frac{1}{2}$ was considered, the arguments carry over. Let $p$ be the probability to go left (corresponding to bit 0) and $q$ the probability to go right (corresponding to bit 1) in a trie, we think about those who go right as the *survivors,* who repeat the experiment. In this way, we always move to the right. And we are searching for an element .11111... (sufficiently many 1's), which is not present in the data structure, in other words we consider the *unsuccessful search cost, followed by an insertion* (which is the cost of inserting this element), provided that we have $n$ random data in the trie. For the symmetric case, this makes perhaps more sense, as we are just

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interested in the parameter *unsuccessful search cost,* as we are no longer considering the path that always goes right, but rather a random path.

Of course, these two special cases are not necessary to understand the paper, but they serve as a *motivation.*

The idea of introducing a probability $\mu$ of escaping the demon is borrowed from [9]; in this thesis U. Schmid studied the collision resolution schemes, related to $n$ transmitted data, using simple tree-algorithms (Capetanakis, Hayes, Tsybakov, Mikhailov). Unlike earlier approaches, Schmid assumes that with a positive probability $\mu$, one of the colliding packages survives and is successfully submitted; compare also [10, 11].

In the following we are interested in the random variable $K$: highest level reached by a party of $n$ players. We are able to compute *all moments* (asymptotically) in an almost automatic fashion. This will be done with the techniques worked out in [5]. Note that the expectated value for the symmetric case $p = q = \frac{1}{2}$ was computed using *Rice's method* in [8].

## 2. Notations

We list for convenience the notations used in this paper.

$$\begin{aligned}
n &:= \text{ number of persons,}\\
\pi(n,k) &:= \mathbb{P}[n \text{ persons reach level } k\text{, but no higher level}], \quad \pi(n,0)=0,\ \pi(0,1)=1,\\
\nu &:= \text{Probability that the demon kills a survivor,} \quad \mu = 1-\nu,\\
q &:= \text{ Probability of flipping `1' and advancing,} \quad p = 1-q,\\
F_n(u) &:= \sum_{k=1}^{\infty} \pi(n,k)u^k, \quad F_0(u) = u,
\end{aligned}$$

the generating polynomial where the coefficient of $u^k$ gives the probability that the party made it exactly to level $k$,

$$\begin{aligned}
G(z,u) &:= \sum_{n=0}^{\infty} F_n(u)\frac{z^n}{n!}, \quad G(0,u) = u,\\
D(z,u) &:= e^{-z}G(z,u) = \sum_{n=0}^{\infty} \frac{z^n}{n!}D_n(u), \quad D(0,u) = u,\\
L &:= \ln 1/q,\\
\log x &= \log_{1/q} x,\\
\widetilde{\alpha} &:= \alpha/L,\\
\chi_l &:= 2l\pi\mathbf{i}/L, \quad l \in \mathbb{Z},\\
\{x\} &:= \text{ fractional part of } x.
\end{aligned}$$

Furthermore, we need a few concepts from $q$-analysis:

$$(x)_n := (1-x)(1-xq)\dots(1-xq^{n-1});$$

often, one writes $(x;q)_n$ to emphasize the parameter $q$, but that is not necessary here. $(x)_\infty := \lim_{n\to\infty}(x)_n$.

Euler's two partition identities:

$$\prod_{l=0}^{\infty}(1-uq^l) = \sum_{t=0}^{\infty}\frac{(-1)^t u^t q^{\binom{t}{2}}}{(q)_t}, \tag{1}$$

$$\prod_{i=1}^{\infty}(1-tq^{i-1})^{-1} = \sum_{u=0}^{\infty}\frac{1}{(q)_u}t^u. \tag{2}$$

They are special cases of Cauchy's formula ($q$-binomial theorem)

$$\frac{(at)_\infty}{(t)_\infty} = \sum_{t=0}^{\infty}\frac{(a)_n t^n}{(q)_n},$$

which we will use later. These concepts can be found in [1].

The following abbreviations will be useful:

$$\begin{aligned} Q_1 &:= (q)_\infty, \\ Q_2 &:= (\mu q)_\infty, \\ H_1(\alpha) &:= (e^\alpha)_\infty, \\ H_2(\alpha) &:= (\mu q e^\alpha)_\infty. \end{aligned}$$

We use the (now standard) notation $[z^n]f(z)$ to extract the coefficient of $z^n$ in the series expansion of $f(z)$.

## 3. Recurrences

We have

$$\pi(n,k) := \sum_{j=1}^{n}\binom{n}{j}q^j p^{n-j}[\nu\pi(j-1,k-1)+\mu\pi(j,k-1)] + p^n[\![k=1]\!],$$

$$F_n(u) = u\sum_{j=1}^{n}\binom{n}{j}q^j p^{n-j}[\nu F_{j-1}(u)+\mu F_j(u)] + up^n, \quad n\ge 1,\ F_0(u)=u.$$

$$G\Big(\frac{z}{p},u\Big) = u\mu e^z G\Big(\frac{zq}{p},u\Big) - u^2\mu e^z + \nu\mu e^z + \nu\sum_{n=1}^{\infty}\frac{z^n}{n!}u\sum_{j=1}^{n}\binom{n}{j}\Big(\frac{q}{p}\Big)^j F_{j-1}(u) + ue^z.$$

Now we differentiate w.r.t. $z$. (The prime notation refers to this.)

$$\begin{aligned} \frac{1}{p}G'\Big(\frac{z}{p},u\Big) &= u\mu e^z G\Big(\frac{zq}{p},u\Big) + \frac{q}{p}u\mu e^z G'\Big(\frac{zq}{p},u\Big) - u^2\mu e^z \\ &\quad + \nu\sum_{n=1}^{\infty}\frac{z^{n-1}}{(n-1)!}u\sum_{j=1}^{n}\binom{n}{j}\Big(\frac{q}{p}\Big)^j F_{j-1}(u) + ue^z. \end{aligned}$$

This translates into

$$\frac{1}{p}D'\Big(\frac{z}{p},u\Big)+\frac{q}{p}D\Big(\frac{z}{p},u\Big)=u\left[\mu\frac{q}{p}D'\Big(\frac{zq}{p},u\Big)+\frac{q}{p}D\Big(\frac{zq}{p},u\Big)\right],$$

or

$$D'(z,u)+qD(z,u)=u\mu qD'(zq,u)+uqD(zq,u).$$

Comparing coefficients, we find

$$D_n(u)=D_{n-1}(u)(uq^n-q)/(1-u\mu q^n),$$

from which we get, upon iteration, the explicit form

$$D_n(u)=u(-1)^nq^n\frac{(u)_n}{(u\mu q)_n}.$$

Since

$$F_n(u)=\sum_{j=0}^{n}\binom{n}{j}D_j(u),$$

we can continue:

$$\begin{aligned}F_n(u)&=u\sum_{j=0}^{n}\binom{n}{j}(-1)^jq^j\frac{(u)_j}{(u\mu q)_j}\\&=u\sum_{j=0}^{n}\binom{n}{j}(-1)^jq^j\frac{(u)_\infty}{(u\mu q)_\infty}\frac{(u\mu q^{j+1})_\infty}{(uq^j)_\infty}\\&=u\frac{(u)_\infty}{(u\mu q)_\infty}\sum_{j=0}^{n}\binom{n}{j}(-1)^jq^j\sum_{k=0}^{\infty}\frac{(uq^j)^k(\mu q)_k}{(q)_k}\\&=u\frac{(u)_\infty}{(u\mu q)_\infty}\sum_{k=0}^{\infty}\frac{u^k}{(q)_k}(\mu q)_k\sum_{j=0}^{n}\binom{n}{j}(-1)^jq^{j(k+1)}\\&=u\frac{(u)_\infty}{(u\mu q)_\infty}\sum_{k=0}^{\infty}\frac{u^k}{(q)_k}(\mu q)_k\big(1-q^{k+1}\big)^n.\end{aligned}$$

Reading off the coefficient $[u^l]F_n(u)$, we get the following explicit result.

**Proposition 1.**

$$\pi(n,l)=\sum_{i+j+h=l-1}\frac{(\mu q)^i}{(q)_i}\frac{(-1)^jq^{\binom{j}{2}}}{(q)_j}\frac{(\mu q)_h}{(q)_h}(1-q^{h+1})^n.$$

Note that the special case $\mu=0$, which restricts the summation to $i=0$, leads to

$$\sum_{j=0}^{l-1}\frac{(-1)^jq^{\binom{j}{2}}}{(q)_j(q)_{l-1-j}}(1-q^{l-j})^n$$

which is exactly Flajolet's formula [2, (46)]. We can even derive a formula with only one summation, again by invoking the $q$-binomial theorem:

$$\pi(n,l)=[u^{l-1}]\frac{(u)_\infty}{(u\mu q)_\infty}\sum_{k=0}^{\infty}\frac{u^k}{(q)_k}(\mu q)_k\big(1-q^{k+1}\big)^n$$

$$= \sum_{k=0}^{l-1} \frac{(\mu q)_k}{(q)_k} \big(1 - q^{k+1}\big)^n [u^{l-1-k}] \frac{(u)_\infty}{(u\mu q)_\infty}$$
$$= \sum_{k=0}^{l-1} \frac{(\mu q)_k}{(q)_k} \big(1 - q^{k+1}\big)^n \frac{(1/(\mu q))_{l-1-k}}{(q)_{l-1-k}} (\mu q)^{l-1-k}.$$

However, we will not use this form; one disadvantage is that for $\mu = 0$, one must consider a limit.

## 4. Asymptotics

Now we set $\eta = l - \log n$ and let $n \to \infty$. This gives, in the range $\eta = \mathcal{O}(1)$, the limiting distribution

$$\pi(n,l) \sim f(\eta) = \frac{Q_2}{Q_1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\mu q)^i}{(q)_i} \frac{(-1)^j q^{\binom{j}{2}}}{(q)_j} \exp(-e^{-L\eta + L(i+j)}),$$

where we recognize the Gumbel distribution function $\exp(-e^{-x})$. To show that the limiting moments are equivalent to the moments of the limiting distribution, we need a suitable rate of convergence (in particular for large and small values of $\eta$). This is related to a uniform integrability condition (see Loève [4, Section 11.4]). For the kind of limiting distribution we consider here, the rate of convergence is analyzed in detail in [5] and [6], we will not repeat the arguments. Set

$$\phi(\alpha) = \int_{-\infty}^{\infty} e^{\alpha\eta} f(\eta) d\eta$$
$$= \frac{Q_2}{Q_1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\mu q)^i}{(q)_i} \frac{(-1)^j q^{\binom{j}{2}}}{(q)_j} e^{\alpha(i+j)} \Gamma(-\widetilde{\alpha})/L = \frac{Q_2}{Q_1} \frac{H_1(\alpha)}{H_2(\alpha)} \Gamma(-\widetilde{\alpha})/L.$$

This function will be the main tool we need to derive all asymptotic moments.

## 5. Moments

We have, with the notations of Louchard and Prodinger [5],

$$\mathbb{E}\left[(K - \log n)^i\right] \sim \widetilde{m}_i + w_i + \mathcal{O}(n^{-\beta}), \quad \beta > 0,$$

where $\widetilde{m}_i$ are constants and $w_i$ are periodic functions of $\log n$, with small $< 10^{-5}$ amplitude. All these expressions *only depend on $\phi(\alpha)$ and its derivatives.* For instance,

$$\phi(0) = 1,$$
$$\widetilde{m}_1 = \phi'(0),$$
$$\widetilde{m}_2 = \phi''(0),$$
$$w_1 = \sum_{l \neq 0} \varphi_1(\chi_l) e^{-2l\pi\mathbf{i}\log n},$$
$$\varphi_1(\chi_l) = \phi'(\alpha)|_{\alpha = -L\chi_l},$$

$$w_2 = \sum_{l \neq 0} \varphi_2(\chi_l) e^{-2l\pi \mathbf{i} \log n},$$
$$\varphi_2(\chi_l) = \phi''(\alpha)|_{\alpha=-L\chi_l},$$

Also note the following local expansions for $\widetilde{\alpha}$ close to 0 resp. $-\chi_l$; recall that $\alpha = \widetilde{\alpha}L$:

$$\Gamma(-\widetilde{\alpha}) = -\frac{L}{\alpha} - \gamma - \frac{\pi^2 + 6\gamma^2}{12L}\alpha + \cdots,$$
$$\Gamma(-\widetilde{\alpha}) = \Gamma(\chi_l) - \frac{\psi(\chi_l)\Gamma(\chi_l)}{L}(\alpha + L\chi_l) + \frac{\Gamma(\chi_l)\big(\psi(1,\chi_l) + \psi^2(\chi_l)\big)}{2L^2}(\alpha + L\chi_l)^2 + \cdots$$

With the appendix identities, this leads to

**Theorem 1.**

$$\widetilde{m}_1 = \frac{2\gamma + L - 2LC_{1,1} + 2L\mu q C_{2,1}}{L},$$
$$\varphi_1(\chi_l) = -\frac{\Gamma(\chi_l)}{L},$$
$$\widetilde{m}_2 = [\pi^2 + 6\gamma^2 + 6\gamma L - 12\gamma L C_{1,1} + 12\gamma L\mu q C_{2,1} + 2L^2 - 12L^2 C_{1,1} - 6L^2 C_{1,2} + 6L^2 C_{1,1}^2 + 12L^2\mu q C_{2,1} + 6L^2\mu^2 q^2 C_{2,2} + 6L^2\mu^2 q^2 C_{2,1}^2 - 12L^2\mu q C_{2,1}C_{1,1}]/(6L^2),$$
$$\varphi_2(\chi_l) = -\frac{(-2\psi(\chi_l) + L - 2LC_{1,1} + 2L\mu q C_{2,1})\Gamma(\chi_l)}{L^2}.$$

The first two expressions are identical to Prodinger[8]. All moments can be automatically obtained by the same method.

For the reader's convenience, we explicitly write the expected value of the maximum level that a party of $n$ people reaches:

$$\mathbb{E}(K) \sim \log_{1/q}(n) + \frac{2\gamma}{L} + 1 - 2\sum_{i=1}^{\infty}\frac{q^i}{1-q^i} + 2\sum_{i=1}^{\infty}\frac{\mu q^i}{1-\mu q^i} + \delta(\log_{1/q}(n))$$

with

$$\delta(x) = -\frac{1}{L}\sum_{l \in \mathbb{Z}\setminus\{0\}} \Gamma(\chi_l)e^{-2\pi \mathbf{i} l x}.$$

## 6. Conclusion

This note is another showcase of the machinery developed in [5]. Once the underlying distribution is Gumbel distributed (extreme value distribution), moments can be computed in a semi-automatic way.

We hope to extend this series of applications in the near future.

# Appendices.

## Appendix A. Identities related to $H_1(\alpha)$

We find it useful to introduce the functions

$$\Sigma_{1,k}(z) := (k-1)! \sum_{i=1}^{\infty} q^{ki}/(1+zq^i)^k.$$

It is easily noticed that

$$\Sigma'_{1,k}(z) = -\Sigma_{1,k+1}(z).$$

The special values

$$C_{1,k} := \sum_{i=1}^{\infty} q^{ki}/(1-q^i)^k = \frac{1}{(k-1)!}\Sigma_{1,k}(-1)$$

are also of interest.

Logarithmic differentiation produces the following formulæ.

$$\begin{aligned}
(-qz)'_\infty &= (-qz)_\infty \Sigma_{1,1},\\
(-qz)''_\infty &= (-qz)_\infty [\Sigma_{1,1}^2 - \Sigma_{1,2}],\\
(-qz)'''_\infty &= (-qz)_\infty [-3\Sigma_{1,1}\Sigma_{1,2} + \Sigma_{1,1}^3 + \Sigma_{1,3}],\\
(-z)'_\infty &= (-qz)_\infty [1 + (1+z)\Sigma_{1,1}],\\
(-z)''_\infty &= (-qz)_\infty [2\Sigma_{1,1} + (1+z)[-\Sigma_{1,2} + \Sigma_{1,1}^2]],\\
(-z)'''_\infty &= (-qz)_\infty [-3\Sigma_{1,2} + 3\Sigma_{1,1}^2 + (1+z)[\Sigma_{1,3} - 3\Sigma_{1,2}\Sigma_{1,1} + \Sigma_{1,1}^3]];
\end{aligned}$$

we wrote here $\Sigma_{1,k}$ for $\Sigma_{1,k}(z)$.

Let $\partial_\alpha$ and $\partial_z$ be the operator that differentiate w.r.t. $\alpha$ resp. $z$. Then we get by the chain rule for any $K(z)$, with $z = -e^\alpha$ or $z = -\mu q e^\alpha$:

$$\begin{aligned}
\partial_\alpha K &= z\partial_z K,\\
\partial_\alpha^2 K &= z[z\partial_z^2 K + \partial_z K],\\
\partial_\alpha^3 K &= z[\partial_z K + 3z\partial_z^2 K + z^2\partial_z^3 K].
\end{aligned}$$

This leads to (recall that $H_1(\alpha) = (e^\alpha)_\infty$)

$$\begin{aligned}
H_{1,0} &:= H_1(0) = 0,\\
H_{1,1} &:= \partial_\alpha H_1(\alpha)|_{\alpha=0} = -Q_1,\\
H_{1,2} &:= \partial_\alpha^2 H_1(\alpha)|_{\alpha=0} = Q_1[-1 + 2C_{1,1}],\\
H_{1,3} &:= \partial_\alpha^3 H_1(\alpha)|_{\alpha=0} = Q_1[-1 + 6C_{1,1} + 3C_{1,2} - 3C_{1,1}^2];
\end{aligned}$$

Note that we obtain the same expressions for $\alpha = -L\chi_l$, as $e^{-L\chi_l} = 1$.

## Appendix B. Identities related to $H_2(\alpha)$

Now we deal with $H_2(\alpha) = (\mu q e^{\alpha})_\infty$.

We need

$$\Sigma_{2,k}(z) := (k-1)! \sum_{i=0}^{\infty} q^{ki}/(1+zq^i)^k = \frac{(k-1)!}{(1+z)^k} + \Sigma_{1,k}(z)$$

and

$$C_{2,k} = \sum_{i=0}^{\infty} q^{ki}/(1-\mu q q^i)^k = \frac{1}{(k-1)!}\Sigma_{2,k}(-\mu q).$$

Since

$$\begin{aligned}
(-z)'_\infty &= (-z)_\infty \Sigma_{2,1}(z), \\
(-z)''_\infty &= (-z)_\infty [\Sigma_{2,1}^2(z) - \Sigma_{2,2}(z)],
\end{aligned}$$

we get

$$\begin{aligned}
H_{2,0} &:= H_2(0) = Q_2, \\
H_{2,1} &:= \partial_\alpha H_2(\alpha)|_{\alpha=0} = -\mu q C_{2,1} Q_2, \\
H_{2,2} &:= \partial_\alpha^2 H_2(\alpha)|_{\alpha=0} = -\mu q Q_2 [C_{2,1} - \mu q(-C_{2,2} + C_{2,1}^2)],
\end{aligned}$$

Again we obtain the same expressions for $\alpha = -L\chi_l$, as $e^{-L\chi_l} = 1$.

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