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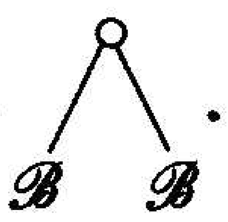
FIBONACCI NUMBERS OF GRAPHS III: PLANTED PLANE TREES

1. INTRODUCTION

In [8], the Fibonacci number $f(G)$ of a (simple) graph G is introduced as the total number of all Fibonacci subsets S of the vertex $V(G)$ of G, where a Fibonacci subset S is a (possibly empty) subset of $V(G)$ such that any two vertices of S are not adjacent. In Graph Theory, a Fibonacci subset is called an independent or internally stable set of vertices.

In [6], the average Fibonacci number of binary trees of size n has been considered for the first time: The family $\mathcal{B}$ of all binary trees is defined by the following equation ($\square$ is the symbol for a leaf and $\circ$ for an internal node), compare [7]:

$\mathcal{B} = \square +$  . (1.1)

Denoting by $h_n(\mathcal{B})$ the total number of Fibonacci subsets of all binary trees of size n, i.e., with n internal nodes, it has been shown in [6] that

$$h_n(\mathcal{B}) \sim (0.63713\ldots)(0.15268\ldots)^{-n} \cdot n^{-3/2}, \tag{1.2}$$

so that the average value $S_n(\mathcal{B})$ of the Fibonacci number of a binary tree of size n fulfils asymptotically

$$S_n(\mathcal{B}) \sim (1.12928\ldots)(1.63742\ldots)^n. \tag{1.3}$$

In Section 2 of the present paper we present an explicit formula for $h_n(\mathcal{B})$ and exact values for the numerical constants in (1.2) and (1.3).

In Section 3 we generalize the foregoing results to the family $\mathcal{T}$ of t-ary trees. Moreover, we determine how the number $h_n(\mathcal{T})$ of all Fibonacci subsets divides up into the numbers $h_{n,j}(\mathcal{T})$ of Fibonacci subsets of cardinality j. The

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order of exponential growth of $S_n(\mathscr{T})$ is analyzed asymptotically for $t \to \infty$.

In the last section we deal with the family $\mathscr{P}$ of all planted plane trees. As a paradigm, we also investigate the second-order moments.

2. BINARY TREES

Let $f_n = f_n(\mathscr{B})$ resp. $g_n = g_n(\mathscr{B})$ be defined by

$$f_n = \sum_{T \in \mathscr{B}_n} \operatorname{card}\{S : S \subseteq V(T);\ S \text{ a Fibonacci subset } \textit{not} \text{ containing the root}\},$$

$$g_n = \sum_{T \in \mathscr{B}_n} \operatorname{card}\{S : S \subseteq V(T);\ S \text{ a Fibonacci subset containing the root}\},$$

where $\mathscr{B}_n$ denotes the family of trees of size n in $\mathscr{B}$. Also let

$$f(z) = \sum_{n \geq 1} f_n z^n \quad \text{and} \quad g(z) = \sum_{n \geq 1} g_n z^n$$

be the corresponding generating functions. From (1.1), we derive the functional equations

$$\begin{aligned} f &= z(1 + f + g)^2 \\ g &= z(1 + f)^2. \end{aligned} \tag{2.1}$$

Substituting $u = z(1 + f)$, we have

$$g = u^2/z, \quad f = u^2(1 + u)^2/z,$$

whence

$$u = z + zf = z + u^2(1 + u)^2 \tag{2.2}$$

or

$$z = \frac{u}{\varphi(u)} \quad \text{with } \varphi(u) = (1 - u(1 + u)^2)^{-1}.$$

Applying Lagrange's inversion formula (LIF) (see [3]), it turns out that the coefficients $[z^n]u(z)$ of z^n in the (formal) power series $u(z)$ fulfil

$$\begin{aligned} [z^{n+1}]u(z) &= \frac{1}{n+1}[w^n](1 - w(1 + w)^2)^{-n-1} \\ &= \frac{1}{n+1}\sum_{j=0}^{n}\binom{n+j}{j}\binom{2j}{n-j}, \end{aligned} \tag{2.3}$$

from which the coefficients of $f(z)$ are immediate. In order to expand $u^2 = zg$, we again apply LIF to get

$$\begin{aligned}[z^{n+1}]u^2(z) &= \frac{2}{n+1}[w^{n-1}](1 - w(1+w)^2)^{-n-1} \\ &= \frac{2}{n+1}\sum_{j=0}^{n-1}\binom{n+j}{j}\binom{2j}{n-1-j}.\end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4), we have

Theorem 1:

$$f_n(\mathcal{B}) = \frac{1}{n+1}\sum_{j=0}^{n}\binom{n+j}{j}\binom{2j}{n-j}$$

$$g_n(\mathcal{B}) = \frac{2}{n+1}\sum_{j=0}^{n-1}\binom{n+j}{j}\binom{2j}{n-1-j}$$

$$h_n(\mathcal{B}) = \frac{1}{n+1}\sum_{j=0}^{n}\frac{j+1}{2j+1}\binom{n+j+1}{j+1}\binom{2j+1}{n-j}$$

$$\begin{aligned}S_n(\mathcal{B}) &= \frac{h_n(\mathcal{B})}{\text{card } \mathcal{B}_n} \\ &= \sum_{j=0}^{n}\frac{j+1}{2j+1}\binom{n+j+1}{j+1}\binom{2j+1}{n-j}\Big/\binom{2n}{n}.\end{aligned}$$

Observing the formulas in the above theorem, the question arises whether there is a combinatorial interpretation of the summands. This will be settled in a more general context in the next section.

Let us now turn to the asymptotic evaluation of the numbers appearing in Theorem 1. The common singularity $\rho = \rho(\mathcal{B})$ nearest to the origin of the generating functions $f(z)$, $g(z)$, $h(z) = 1 + f(z) + g(z)$ has been determined numerically in [6], $\rho = 0.15268\ldots$ [compare (1.1)]. In the following, we will give the exact value of this constant, i.e., the singularity nearest to the origin of the function $u(z)$ from above.

By (2.2), ρ is a solution z of the system

$$\begin{aligned}H(z, u) &= u^2(1+u)^2 - u + z = 0 \\ \frac{\partial H}{\partial u}(z, u) &= 4u^3 + 6u^2 + 2u - 1 = 0\end{aligned} \tag{2.5}$$

(Darboux's method; compare [1], [4], and [5]). Solving the second equation for u by Cardano's formula and inserting into the first equation, it turns out that

$$\rho = -y^4 + \frac{1}{2}y^2 + y - \frac{9}{16} \tag{2.6}$$

with

$$y = \frac{1}{2}\left(\sqrt[3]{1 + \sqrt{\frac{26}{27}}} + \sqrt[3]{1 - \sqrt{\frac{26}{27}}}\right).$$

Again following Darboux's method cited above, we obtain

$$[z^n]u(z) \sim c \cdot \rho^{-n} \cdot n^{-3/2}, \quad n \to \infty, \tag{2.7}$$

with

$$c = \left(\frac{\rho}{2\pi(-1 + 12y^2)}\right)^{1/2}.$$

Hence

$$f_n = [z^{n+1}]u(z) \sim \frac{c}{\rho}\rho^{-n}n^{-3/2}. \tag{2.8}$$

To determine the asymptotic behavior of g_n, we observe that

$$u(z) = u(\rho) - K(\rho - z)^{1/2} + \cdots;$$

thus,

$$u^2(z) = u^2(\rho) - 2u(\rho)K(\rho - z)^{1/2} + \cdots,$$

so that

$$g_n = [z^{n+1}]u^2(z) \sim 2u(\rho)f_n = 2\left(y - \frac{1}{2}\right)f_n. \tag{2.9}$$

Putting everything together, we arrive at

Theorem 2: With

$$y = \frac{1}{2}\left(\sqrt[3]{1 + \sqrt{\frac{26}{27}}} + \sqrt[3]{1 - \sqrt{\frac{26}{27}}}\right)$$

and

$$\rho = -y^4 + \frac{1}{2}y^2 + y - \frac{9}{16},$$

we have

$$f_n(\mathscr{B}) \sim \sqrt{\frac{1}{2\rho\pi(-1 + 12y^2)}} \cdot \rho^{-n}n^{-3/2}$$

$$\sim (0.41878180\ldots) \cdot (0.15267965\ldots)^{-n}n^{-3/2},$$

(continued)

$$g_n(\mathscr{B}) \sim 2\left(y - \frac{1}{2}\right)f_n(\mathscr{B})$$

$$\sim (0.21834433\ldots)(0.15267965\ldots)^{-n}n^{-3/2},$$

$$h_n(\mathscr{B}) \sim 2yf_n(\mathscr{B}) \sim \sqrt{\frac{2y^2}{\rho\pi(-1 + 12y^2)}} \cdot \rho^{-n}n^{-3/2}$$

$$\sim (0.63712614\ldots)(0.15267965\ldots)^{-n}n^{-3/2}.$$

In particular,

$$\frac{f_n}{g_n} \sim \frac{1}{2y - 1} = 1.917987\ldots,$$

$$S_n(\mathscr{B}) = \frac{h_n(\mathscr{B})}{\text{card } \mathscr{B}_n} \sim \sqrt{\frac{2y^2}{\rho(-1 + 12y^2)}} \cdot \left(\frac{1}{4\rho}\right)^{-n}$$

$$\sim (1.1292766\ldots)(1.6374152\ldots)^n.$$

3. t-ARY TREES

As announced in Section 2, we now determine the numbers

$$f_{n,j} = f_{n,j}(\mathscr{T}) = \sum_{T \in \mathscr{T}_n} \text{card}\{S : S \subseteq V(T);\ S \text{ a Fibonacci subset of cardinality } j \text{ } \textit{not} \text{ containing the root}\},$$

and

$$g_{n,j} = g_{n,j}(\mathscr{T}) = \sum_{T \in \mathscr{T}_n} \text{card}\{S : S \subseteq V(T);\ S \text{ a Fibonacci subset of cardinality } j \text{ containing the root}\},$$

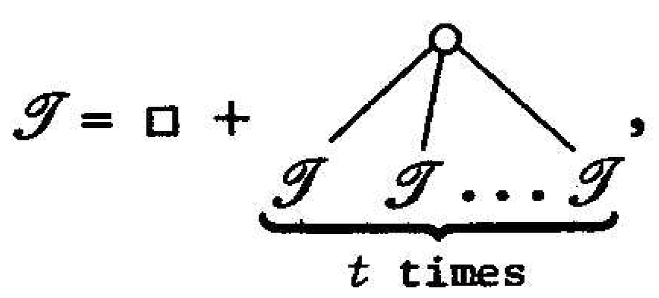
where $\mathscr{T}_n$ denotes the family of t-ary trees of size n. Let

$$F(z, x) = \sum_{n,k} f_{n,k} z^n x^k$$

resp.

$$G(z, x) = \sum_{n,k} g_{n,k} z^n x^k$$

be the double generating functions. Since



, (3.1)

it follows that

$$F = z(1 + F + G)^t$$
$$G = xz(1 + F)^t. \tag{3.2}$$

Substituting $xz = w^{t-1}$ and $V = w(1 + F)$, we have

$$G = \frac{V^t}{w}$$

and

$$F = z\left(\frac{V}{w} + \frac{V^t}{w}\right)^t = \frac{1}{xw} V^t(1 + V^{t-1})^t,$$

so that

$$V = w + wF = w + \frac{1}{x} V^t(1 + V^{t-1})^t$$

and, finally,

$$w = V\left(1 - \frac{1}{x} V^{t-1}(1 + V^{t-1})^t\right). \tag{3.3}$$

Applying LIF

$$V = \sum_k v_k(x)w^k$$

with

$$v_{k+1}(x) = \frac{1}{k+1}[y^k]\left(1 - \frac{1}{x} y^{t-1}(1 + y^{t-1})^t\right)^{-k-1}. \tag{3.4}$$

Since

$$1 + F = \frac{V}{w} = \sum_k v_{k+1}(x)w^k$$
$$= 1 + \sum_{n,k} f_{n,k} x^k z^n$$
$$= 1 + \sum_{n,k} f_{n,k} \left(\frac{1}{x}\right)^{n-k} w^{(t-1)n},$$

we have

$$\sum_{n,k} f_{n,k} \left(\frac{1}{x}\right)^{n-k} = v_{(t-1)n+1}(x)$$
$$= \frac{1}{1 + (t-1)}[y^{(t-1)n}]$$
$$\left(1 - \frac{1}{x} y^{t-1}(1 + y^{t-1})^t\right)^{-(t-1)n-1}$$

$$= \frac{1}{1 + (t - 1)n}[y^n]\left(1 - \frac{1}{x}\, y(1 + y)^t\right)^{-(t-1)n-1}$$

$$= \frac{1}{1 + (t - 1)n} \sum_{j=0}^{n} \binom{(t - 1)n + j}{j}\binom{tj}{n - j}\left(\frac{1}{x}\right)^t,$$

so that

$$f_{n,\,n-j} = \frac{1}{1 + (t - 1)n}\binom{(t - 1)n + j}{j}\binom{tj}{n - j}. \qquad (3.5)$$

In order to investigate $G = V^t/w$, we again use LIF to find that

$$V^t = \sum_k \tilde{v}_k(x) w^k$$

with

$$\tilde{v}_{k+1}(x) = \frac{t}{k + 1}[y^{k+1-t}]\left(1 - \frac{1}{x}\, y^{t-1}(1 + y^{t-1})^t\right)^{-k-1}.$$

By a similar computation,

$$g_{n,\,n-j} = \frac{t}{1 + (t - 1)n}\binom{(t - 1)n + j}{j}\binom{tj}{n - j - 1}. \qquad (3.6)$$

Theorem 3: The average values of the numbers of Fibonacci subsets of cardinality $n - j$ of the trees in $\mathscr{T}_n$ are given by

(a) (not containing the root)

$$\binom{(t - 1)n + j}{j}\binom{tj}{n - j}\Big/\binom{tn}{n};$$

(b) (containing the root)

$$t\binom{(t - 1)n + j}{j}\binom{tj}{n - j - 1}\Big/\binom{tn}{n};$$

(c) (in total)

$$\frac{j + 1}{tj + 1}\binom{(t - 1)n + j + 1}{j + 1}\binom{tj + 1}{n - j}\Big/\binom{tn}{n}.$$

Observe that for $t = 2$ these expressions coincide with the summands in Theorem 1, which means that the desired combinatorial interpretation may be established in this way.

Summing up over all possible values of j, we obtain the following corollary.

Corollary 1: The average Fibonacci number $S_n(\mathscr{T})$ of t-ary trees of size n is given by

$$S_n(\mathscr{T}) = \sum_{j=0}^{n} \frac{j+1}{tj+1}\binom{(t-1)n+j+1}{j+1}\binom{tj+1}{n-j}\Big/\binom{tn}{n}.$$

Before exploring the asymptotic behavior of $S_n(\mathscr{T})$ for $n \to \infty$, we want to stress the question for which value of

$$\alpha \in \left]\frac{1}{t+1}, 1\right[$$

the expression

$$h_{n,\,(1-\alpha)n}(\mathscr{T}) = \frac{\alpha n+1}{t\alpha n+1}\binom{(t-1)n+\alpha n+1}{\alpha n+1}\binom{t\alpha n+1}{n-\alpha n}$$

[compare (c) of Theorem 3] obtains its maximum for $n \to \infty$. By Stirling's approximation, we find

$$h_{n,\,(1-\alpha)n}(\mathscr{T}) \sim \frac{1}{2\pi n}\sqrt{\frac{(t-1+\alpha)^3 t}{(t-1)(1-\alpha)(\alpha(t+1)-1)^3}}\, C_{\alpha,t}^{n},$$

with

$$C_{\alpha,t} = \frac{(t-1+\alpha)^{t-1+\alpha}(t\alpha)^{t\alpha}}{(t-1)^{t-1}\alpha^{\alpha}(1-\alpha)^{1-\alpha}(\alpha(t+1)-1)^{\alpha(t+1)-1}}.$$

Since $C_{\alpha,t}$ regulates the exponential growth, we confine our considerations to this quantity. Let α_t denote the value of α for which $C_{\alpha,t}$ takes its maximum. By ordinary calculus, we find that α_t must fulfil the equation

$$\frac{(\alpha_t + t - 1)\cdot t^t \cdot \alpha_t^{t-1}\cdot(1-\alpha_t)}{(\alpha_t(t+1)-1)^{t+1}} = 1. \tag{3.8}$$

For example,

$$\alpha_1 = \frac{5+\sqrt{5}}{10} = 0.7236067\ldots,$$

$$\alpha_2 = 0.7074302\ldots\,.$$

It is not difficult to see that $\alpha_\infty = \lim_{t\to\infty} \alpha_t$ exists.

Taking the logarithm in (3.8) and expanding for $t \to \infty$, it turns out that α_∞ is the (unique) solution of the equation

$$\frac{1 - \alpha_\infty}{\alpha_\infty^2} = e^{\frac{\alpha_\infty - 1}{\alpha_\infty}} \tag{3.9}$$

with $0 < \alpha_\infty < 1$, i.e.,

$$\alpha_\infty = 0.6924583\ldots\ . \tag{3.10}$$

By a more careful consideration, it turns out that

$$\alpha_t = \alpha_\infty + \frac{\beta}{t} + \mathcal{O}\left(\frac{1}{t^2}\right),\quad t \to \infty.$$

In a similar way as in the determination of α_∞, we find that β is given by the equation

$$\beta\left(\frac{1}{\alpha_\infty^2} + \frac{2}{\alpha_\infty} + \frac{1}{1 - \alpha_\infty}\right) = \alpha_\infty - \frac{3}{2} + \frac{1}{2\alpha_\infty^2}, \tag{3.11}$$

i.e.,

$$\beta = 0.0285962\ldots\ .$$

Altogether, we have proved

Theorem 4: For "large n" the maximal contribution to the average Fibonacci number $S_n(\mathcal{T})$ occurs for a cardinality $j = \gamma_t \cdot n$ of the Fibonacci subsets, where

$$\gamma_t = 1 - \alpha_t = 0.3075416\ldots - \frac{0.0285962\ldots}{t} + \mathcal{O}(1/t^2).$$

To speak in a less rigorous way, we may say that Fibonacci subsets which contain approximately 30% of the nodes of the tree constitute the maximal contribution to the Fibonacci number.

The last part of this section is devoted to the study of the asymptotic behavior of $S_n(\mathcal{T})$ for $n \to \infty$. For this reason, we introduce the generating functions

$$\begin{aligned} f(z) &= F(z, 1) = \sum_n z^n \sum_k f_{n,k}, \\ g(z) &= G(z, 1) = \sum_n z^n \sum_k g_{n,k}. \end{aligned} \tag{3.12}$$

From (3.2), we find that

$$f = z(1 + f + g)^t \quad \text{and} \quad g = z(1 + f)^t. \tag{3.13}$$

Substituting

$$u = z(1 + f)^{t-1}, \tag{3.14}$$

it turns out that

$$g = u^{1+1/(t-1)} \cdot z^{-1/(t-1)}$$

and

$$f = z\left(\frac{u^{1/(t-1)}}{z^{1/(t-1)}} + \frac{u^{1+1/(t-1)}}{z^{1/(t-1)}}\right)^t$$

$$= z^{-1/(t-1)} u^{t/(t-1)} (1 + u)^t .$$

Inserting into (3.14) yields, after a few steps,

$$z = u(1 - u(1 + u)^t)^{t-1}. \tag{3.15}$$

In order to apply Darboux's method, we solve the system

$$\begin{aligned} H(z, u) &= z - u(1 - u(1 + u)^t)^{t-1} = 0 \\ \frac{\partial H}{\partial u}(z, u) &= 0. \end{aligned} \tag{3.16}$$

Let (z_t, u_t) be the pair of solutions in question. Then, after some short manipulations, the second equation (3.16) may be written as

$$tu_t(1 + tu_t)(1 + u_t)^{t-1} = 1. \tag{3.17}$$

From this identity, we gain the asymptotic behavior of u_t for $t \to \infty$ as follows: It is easily seen that $tu_t = \mathcal{O}(1)$. We put

$$tu_t = \delta + r_t . \tag{3.18}$$

Inserting and expanding, we derive

$$(1 + u_t)^{t-1} = e^{\delta}(1 + o(1)), \quad t \to \infty,$$

so that $\delta > 0$ is the (unique) solution of

$$\delta(1 + \delta)e^{\delta} = 1, \text{ i.e., } \delta = 0.4441302\ldots . \tag{3.19}$$

Again plugging (3.18) into (3.19), a more detailed expansion yields

$$(1 + u_t)^{t-1} = e^{\delta}\left(1 + r_t - \frac{\delta}{t} - \frac{\delta^2}{2t} + \mathcal{O}\left(\frac{1}{t}\right) + \mathcal{O}(r_t^2)\right)$$

and therefore

$$r_t = \frac{2\delta + \delta^2}{1 + 3\delta + \delta^2} \cdot \frac{\delta(\delta + 1)}{2} \cdot \frac{1}{t} + \cdots, \quad (t \to \infty),$$

so that

$$u_t = \frac{\delta}{t} + \frac{\varepsilon}{t^2} + \mathcal{O}\left(\frac{1}{t^3}\right) \tag{3.20}$$

with

$$\varepsilon = \frac{\delta^2(\delta + 1)(\delta + 2)}{2((\delta + 1)(\delta + 2) - 1)} = 0.1376138\ldots .$$

Turning now to z_t, (3.15) combined with (3.17) yields

$$\begin{aligned} z_t &= u_t\left(1 - \frac{1 + u_t}{t(1 + tu_t)}\right)^{t-1} \\ &= u_t\left(\frac{t - 1}{t}\right)^{t-1}\left(1 + \frac{u_t}{1 + tu_t}\right)^{t-1} \\ &= tu_t \cdot q_t\left(1 + \frac{u_t}{1 + tu_t}\right)^{t-1}, \end{aligned} \tag{3.21}$$

where

$$q_t = \frac{1}{t}\left(\frac{t - 1}{t}\right)^{t-1} \tag{3.22}$$

is the unique singularity nearest to the origin of the generating function

$$y(z) = \sum_{n \geqslant 0} \frac{1}{1 + (t - 1)n}\binom{tn}{n} z^n$$

of the numbers of trees in $\mathcal{T}_n$.

By Darboux's theorem, it follows that $S_n(\mathcal{T})$ behaves like

$$S_n(\mathcal{T}) \sim A_t\left(\frac{q_t}{z_t}\right)^n, \quad n \to \infty, \tag{3.23}$$

where A_t is a constant that will not be determined explicitly here, for shortness. The ratio q_t/z_t [i.e., the order of growth of $S_n(\mathcal{T})$] behaves for $t \to \infty$, by (3.20) and (3.21), as

$$\frac{q_t}{z_t} = (\delta + 1)e^{\delta^2/(\delta+1)}\left(1 + \frac{1}{t}\left(\frac{\delta}{\delta + 1} + \frac{\delta^2}{2(\delta + 1)^2} - \frac{\varepsilon}{(\delta + 1)^2} - \frac{\varepsilon}{\delta}\right) + \cdots\right).$$

Evaluating the appearing constants numerically, we get

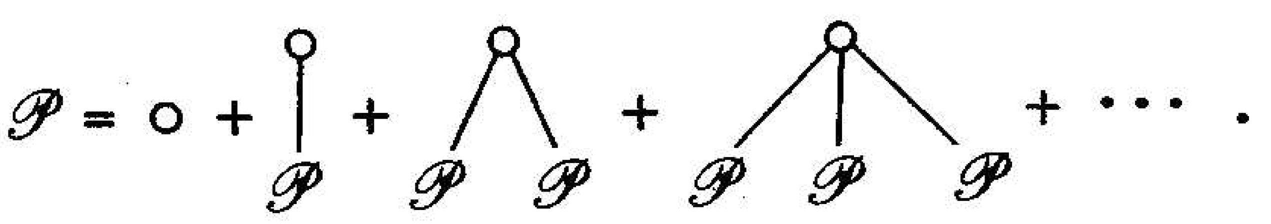
Theorem 5: With A_t a constant, we have

$$S_n(\mathcal{T}) \sim A_t\left(1.655487\ldots - \frac{0.0489690\ldots}{t} + \mathcal{O}\left(\frac{1}{t^2}\right)\right)$$

for $n \to \infty$.

4. PLANTED PLANE TREES

The family $\mathcal{P}$ of planted plane trees is defined by the following symbolic equation:



(4.1)

Let us denote by $f_{n,j} = f_{n,j}(\mathcal{P})$, $g_{n,j} = g_{n,j}(\mathcal{P})$ the numbers of Fibonacci subsets of cardinality j of the trees of size n in $\mathcal{P}$ (not containing resp. containing the root) and by $F(z, x)$ resp. $G(z, x)$ the double generating functions. From (4.1), we obtain

$$F = \frac{z}{1 - F - G}, \quad G = \frac{zx}{1 - F}. \tag{4.2}$$

From this

$$z = \frac{F(1 - F)^2}{1 + F(x - 1)} \tag{4.3}$$

Applying LIF as in the previous section, we obtain

$$\begin{aligned} f_{n,j} &= \frac{1}{n}\binom{n}{j}\binom{2n-2}{n-j-1} \\ g_{n,j} &= \frac{1}{n-1}\binom{n-1}{j-1}\binom{2n-2}{n-j-1}. \end{aligned} \tag{4.4}$$

Theorem 6: The average numbers of Fibonacci subsets of cardinality j of planted plane trees of size n are given by:

(a) (not containing the root)

$$\binom{n}{j}\binom{2n-2}{n-j-1}\Big/\binom{2n-2}{n-1};$$

(b) (containing the root)

$$\frac{n}{n-1}\binom{n-1}{j-1}\binom{2n-2}{n-j-1}\Big/\binom{2n-2}{n-1};$$

(c) (in total)

$$2\binom{n}{j}\binom{2n-3}{n-j-1}\Big/\binom{2n-2}{n-1}.$$

Applying Vandermonde's convolution, we obtain

Corollary 2: The average numbers of Fibonacci subsets of planted plane trees of size n are given by:

(a) (not containing the root)

$$a_n := \binom{3n-2}{n-1}\Big/\binom{2n-2}{n-1};$$

(b) (containing the root)

$$b_n := \frac{n}{n-1}\binom{3n-3}{n-2}\Big/\binom{2n-2}{n-1};$$

(c) (in total)

$$2\binom{3n-3}{n-1}\Big/\binom{2n-2}{n-1} \sim \sqrt{3}\cdot\left(\frac{27}{16}\right)^{n-1},\quad (n\to\infty);$$

(d) $\dfrac{a_n}{b_n} = 3 - \dfrac{2}{n}$.

The second-order moments of all random variables in question are not much harder to obtain than the expected values. To give an example, we determine the second-order moment in the case of planted plane trees.

Let $f(T)$ resp. $g(T)$ denote the number of Fibonacci subsets of the tree T not containing resp. containing the root and

$$A(z) = \sum_n z^n \sum_{T\in\mathcal{P}_n} (f(T) + g(T))^2; \tag{4.4}$$

$$B(z) = \sum_n z^n \sum_{T\in\mathcal{P}_n} f^2(T);$$

(continued)

$$C(z) = \sum_n z^n \sum_{T \in \mathcal{P}_n} f(T)g(T); \tag{4.4}$$

$$D(z) = \sum_n z^n \sum_{T \in \mathcal{P}_n} g^2(T).$$

So we have

$$A = B + 2C + D, \tag{4.5}$$

and, by (4.1),

$$B = z/(1 - A),$$
$$C = z/(1 - B - C), \tag{4.6}$$
$$D = z/(1 - B).$$

From (4.6), it follows that

$$B = \frac{z}{1 - B - 2C - \frac{z}{1 - B}},$$

or

$$z = B(1 - B)(1 - B - 2C),$$

whence

$$2C = 1 - B - \frac{z}{B(1 - B)}.$$

Inserting this into

$$4z = 2C(2(1 - B) - 2C),$$

we derive

$$4z = (1 - B)^2 - \frac{z^2}{B^2(1 - B)^2},$$

or

$$z = \frac{B}{\varphi(B)} \quad \text{with } \varphi(B) = (1 - B)^{-2}(-2B + \sqrt{1 + 4B^2})^{-1}. \tag{4.7}$$

Applying LIF,

$$[z^n]B = \frac{1}{n}[z^{n-1}](1 - z)^{-2n}(2z + \sqrt{1 + 4z^2})^n.$$

Substituting $z = u/(1 - u^2)$, it follows by formal residue calculation that

$$[z^n]B = \frac{1}{n}[u^{n-1}]\frac{(1+u^2)(1+u)^{4n-2}(1-u)^{2n-2}}{(1-u-u^2)^{2n}}, \tag{4.8}$$

whence

$$[z^n]B = \frac{1}{n}\sum_{i+\ell+j=n-1}\binom{2n+i-1}{2n-1}\binom{n-j}{n-\ell-j}\left[\binom{n+\ell+j-1}{j}+\binom{n+\ell+j-1}{j-2}\right]. \tag{4.9}$$

Similarly,

$$[z^n] = \frac{1}{n}[u^{n-1}]\frac{(1-3u)(1+u)^{4n-3}(1-u)^{2n-2}}{(1-u-u^2)^{2n}}, \tag{4.10}$$

$$[z^n] = \frac{1}{n}[u^{n-1}]\frac{(1+u^2)(1-2u-2u^2-2u^3-u^4)(1+u)^{4n-6}(1-u)^{2n-2}}{(1-u-u^2)^{2n}} \tag{4.11}$$

(and $A = B + 2C + D!$).

To perform the asymptotics of $[z^n]A$, we again use Darboux's method. Starting from (4.7), the method already described in the previous sections leads to the numerical value

$$q = 0.08738321\ldots$$

for the singularity q of B (and also C, D, A) nearest to the origin. By local expansions of the generating functions about the singularity q, a tedious computation leads to (compare [5])

Theorem 7:

$$[z^n]A \sim \frac{1.755746\ldots}{2\sqrt{\pi}}\; q^{-n+1/2}\, n^{-3/2}$$

and the second-order moment of the number of Fibonacci subsets is asymptotically given by

$$\frac{[z^n]A}{\text{card}\ \mathscr{P}_n} \sim (1.038020\ldots)(2.860961\ldots)^n.$$

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